On Dual Phase-Space Relativity, the Machian Principle and Modified Newtonian Dynamics

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We investigate the consequences of the Mach's principle of inertia within the context of the Dual Phase Space Relativity which is compatible with the Eddington-Dirac large numbers coincidences and may provide with a physical reason behind the observed anomalous Pioneer acceleration and a solution to the riddle of the cosmological constant problem. The cosmological implications of Non-Archimedean Geometry by assigning an upper impossible scale in Nature and the cosmological variations of the fundamental constants are also discussed. We study the corrections to Newtonian dynamics resulting from the Dual Phase Space Relativity by analyzing the behavior of a test particle in a *modified* Schwarzschild geometry (due to the the effects of the maximal acceleration) that leads in the weak-field approximation to essential *modifications* of the Newtonian dynamics and to violations of the equivalence principle. Finally we follow another avenue and find modified Newtonian dynamics induced by the Yang's Noncommutative Spacetime algebra involving a lower and upper scale in Nature.

1 Introduction

In recent years we have argued that the underlying fundamental physical principle behind string theory, not unlike the principle of equivalence and general covariance in Einstein's general relativity, might well be related to the existence of an invariant minimal length scale (Planck scale) attainable in nature. A scale relativistic theory involving spacetime *resolutions* was developed long ago by Nottale where the Planck scale was postulated as the minimum observer independent invariant resolution in Nature [2]. Since "points" cannot be observed physically with an ultimate resolution, they are fuzzy and smeared out into fuzzy balls of Planck radius of arbitrary dimension. For this reason one must construct a theory that includes all dimensions (and signatures) on the equal footing. Because the notion of dimension is a topological invariant, and the concept of a fixed dimension is lost due to the fuzzy nature of points, dimensions are resolution-dependent, one must also include a theory with *all* topologies as well. It turned out that Clifford algebras contained the appropriate algebro-geometric features to implement this principle of polydimensional transformations that reshuffle a five-brane history for a membrane history, for example. For an extensive review of this Extended Relativity Theory in Clifford Spaces that encompasses the unified dynamics of all p-branes, for different values of the dimensions of the extended objects, and numerous physical consequences, see [1].

A Clifford-space dynamical derivation of the stringy-minimal length uncertainty relations [11] was furnished in [45]. The dynamical consequences of the minimal-length in Newtonian dynamics have been recently reviewed by [44]. The idea of minimal length (the Planck scale L_P) can be incorporated within the context of the maximal acceleration Relativity principle [68] $a_{max} = c^2/L_P$ in Finsler Geometries [56] and [14]. A different approach than the one based on Finsler Geometries is the pseudo-complex Lorentz group description by Schuller [61] related to the effects of maximal acceleration in Born-Infeld models that also maintains Lorentz invariance, in contrast to the approaches of Double Special Relativity (DSR) [70] where the Lorentz symmetry is deformed. Quantum group deformations of the Poincaré symmetry and of Gravity have been analyzed by [69] where the deformation parameter q could be interpreted in terms of an upper and lower scale as $q = e^{L_P/R}$ such that the undeformed limit $q = 1$ can be attained when $L_P \to 0$ and/or when $R \to \infty$ [68]. For a discussions on the open problems of Double Special Relativity theories based on kappa-deformed Poincaré symmetries [63] and motivated by the anomalous Lorentz-violating dispersion relations in the ultra high energy cosmic rays [71, 73], we refer to [70].

An upper limit on the maximal acceleration of particles was proposed long ago by Caianiello [52]. This idea is a direct consequence of a suggestion made years earlier by Max Born on a Dual Relativity principle operating in Phase Spaces [49], [74] where there is an upper bound on the four-force (maximal string tension or tidal forces in strings) acting on a particle as well as an upper bound in the particle's velocity given by the speed of light. For a recent status of the geometries behind maximal-acceleration see [73]; its relation to the Double Special Relativity programs was studied by [55] and the possibility that Moyal deformations of Poincaré algebras could be related to the kappa-deformed Poincaré algebras was raised in [68]. A thorough study of Finsler geometry and Clifford algebras has been undertaken by Vacaru [81] where Clifford/spinor structures were defined with respect to Nonlinear connections associated with certain non-holonomic modifications of Riemann-Cartan gravity. The study of non-holonomic Clifford-Structures in the construction of a Noncommutative Riemann-Finsler Geometry has recently been advanced by [81].

Other implications of the maximal acceleration principle in Nature, like neutrino oscillations and other phenomena, have been studied by [54], [67], [22]. Recently, the variations of the fine structure constant α [64] with the cosmological accelerated expansion of the Universe was recast as a renormalization group-like equation governing the cosmological red shift (Universe scale) variations of α based on this maximal acceleration principle in Nature [68]. The fine structure constant was smaller in the past. Pushing the cutoff scale to the minimum Planck scale led to the intriguing result that the fine structure constant could have been extremely small (zero) in the early Universe and that all matter in the Universe could have emerged via the Unruh-Rindler-Hawking effect (creation of radiation/matter) due to the acceleration w. r. t the vacuum frame of reference. For reviews on the alleged variations of the fundamental constants in Nature see [65].

The outline of this work goes as follows. In section 2 we review the Dual Phase Space Relativity and show why the Planck areas are invariant under acceleration-boosts transformations.

In 3.1 we investigate the consequences of the Mach's principle of inertia within the context of the Dual Phase Space Relativity Principle which is compatible with the Eddington-Dirac large numbers coincidence and may provide with a very plausible physical reason behind the observed anomalous Pioneer acceleration due to the fact that the universe is in accelerated motion (a non-inertial frame of reference) w. r. t the vacuum. Our proposal shares similarities with the previous work of [6], [3]. To our knowledge, the first person who *predicted* the Pioneer anomaly in 1978 was P. LaViolette [5], from an entirely different approach based on the novel theory of sub-quantum kinetics to explain the vacuum fluctuations, two years *prior* to the Anderson et al observations [7]. The cosmological implications of Non-Archimedean Geometry [94] by assigning an upper impassible scale in Nature [2] and the cosmological variations of the fundamental constants are also discussed.

In 3.2 the crucial modifications to Newtonian dynamics resulting from the Dual Phase Space Relativity are analyzed further. In particular, the physical consequences of an upper and lower bounds in the acceleration and an upper and lower bounds in the angular velocity. We study the particular behavior of a test particle living in a *modified* Schwarzschild geometry (due to the the effects of the principle of maximal acceleration) that leads in the weak-field approximation to essential *modifications* of the Newtonian dynamics and to violations of the equivalence principle. For violations of the equivalence principle in neutrino oscillations see [42], [54].

Finally, in 4 we study another interesting avenue for the origins of modified Newtonian dynamics based on Yang's Noncommutative Spacetime algebra involving a lower and upper scale [136] that has been revisited recently by us [134] in the context of holography and area-quantization in C-spaces (Clifford spaces); in the physics of D-branes and covariant Matrix models by [137] and within the context of Lie algebra stability by [48]. A different algebra with two length scales has been studied by [43] in order to account for modifications of Newtonian dynamics (that also violates the equivalence principle).

2 Dual Phase-Space Relativity

In this section we will review in detail the Born's Dual Phase Space Relativity and the principle of Maximal-acceleration Relativity [68] from the perspective of $8D$ Phase Spaces and the role of the invariance $U(1,3)$ Group. We will focus for simplicity on a *flat* $8D$ Phase Space. A *curved* case scenario has been analyzed by Brandt [56] within the context of the Finsler geometry of the $8D$ tangent bundle of spacetime and written the generalized $8D$ gravitational equations that reduce to the ordinary Einstein-Riemannian gravitational equations in the *infinite* acceleration limit. Vacaru [81] has constructed the Riemann-Finsler geometries endowed with non-holonomic structures induced by *nonlinear* connections and developed the formalism to build a Noncommutative Riemann-Finsler Geometry by introducing suitable Clifford structures. A curved *momentum* space geometry was studied by [50]. Toller [73] has explored the different possible geometries associated with the maximal acceleration principle and the physical implications of the meaning of an "observer", "measuring device" in the cotangent space.

The $U(1,3) = SU(1,3) \otimes U(1)$ Group transformations, which leave invariant the phase-space intervals under rotations, velocity and acceleration boosts, were found by Low [74] and can be simplified drastically when the velocity/acceleration boosts are taken to lie in the z-direction, leaving the transverse directions x, y, p_x, p_y intact; i. e., the $U(1,1) = SU(1,1) \otimes U(1)$ subgroup transformations that leave invariant the phase-space interval are given by (in units of $\hbar = c = 1$)

$$\begin{aligned}(d\omega)^2 &= (dT)^2 - (dX)^2 + \frac{(dE)^2 - (dP)^2}{b^2} = \\ &= (d\tau)^2 \left[1 + \frac{(dE/d\tau)^2 - (dP/d\tau)^2}{b^2}\right] = \\ &= (d\tau)^2 \left[1 - \frac{m^2 g^2(\tau)}{m_P^2 A_{max}^2}\right],\end{aligned} \tag{2.1}$$

where we have factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in eq-(2.1) and the maximal proper-force is set to be $b \equiv m_P A_{max}$. Here m_P is the Planck mass $1/L_P$ so that $b = (1/L_P)^2$, may also be interpreted as the maximal string tension when L_P is the Planck scale.

The quantity $g(\tau)$ is the proper four-acceleration of a particle of mass m in the z-direction which we take to be defined by the X coordinate. The interval $(d\omega)^2$ described by Low [74] is $U(1,3)$-invariant for the most general transformations in the $8D$ phase-space. These transformations are rather elaborate, so we refer to the references [74] for details. The appearance of the $U(1,3)$ group in $8D$ Phase Space is not too surprising since it could be seen as the "complex doubling" version of the Lorentz group $SO(1,3)$. Low discussed the irreducible unitary representations of such $U(1,3)$ group and the relevance for the strong interactions of quarks and hadrons since $U(1,3)$, with 16 generators, contains the $SU(3)$ group.

The analog of the Lorentz relativistic factor in eq-(2.1) involves the ratios of two proper *forces*. One variable force is given by $mg(\tau)$ and the maximal proper force sustained by an *elementary* particle of mass m_P (a *Planckton*) is assumed to be $F_{max} = m_{Planck} c^2 / L_P$. When $m = m_P$, the ratio-squared of the forces appearing in the relativistic factor of eq-(2.1) becomes then g^2/A^2_{max}, and the phase space interval coincides with the geometric interval discussed by [61], [54], [67], [22].

The transformations laws of the coordinates in that leave invariant the interval (2.1) were given by [74]:

$$T' = T\cosh\xi + \left(\frac{\xi_v X}{c^2} + \frac{\xi_a P}{b^2}\right)\frac{\sinh\xi}{\xi}, \qquad (2.2a)$$

$$E' = E\cosh\xi + (-\xi_a X + \xi_v P)\,\frac{\sinh\xi}{\xi}, \qquad (2.2b)$$

$$X' = X\cosh\xi + \left(\xi_v T - \frac{\xi_a E}{b^2}\right)\frac{\sinh\xi}{\xi}, \qquad (2.2c)$$

$$P' = P\cosh\xi + \left(\frac{\xi_v E}{c^2} + \xi_a T\right)\frac{\sinh\xi}{\xi}. \qquad (2.2d)$$

The ξ_v is velocity-boost rapidity parameter and the ξ_a is the force/acceleration-boost rapidity parameter of the primed-reference frame. They are defined respectively:

$$\tanh\left(\frac{\xi_v}{c}\right) = \frac{v}{c}, \quad \tanh\left(\frac{\xi_a}{b}\right) = \frac{ma}{m_P A_{max}}. \qquad (2.3)$$

The *effective* boost parameter ξ of the $U(1,1)$ subgroup transformations appearing in eqs-(2.2a, 2.2d) is defined in terms of the velocity and acceleration boosts parameters ξ_v, ξ_a respectively as:

$$\xi \equiv \sqrt{\frac{\xi_v^2}{c^2} + \frac{\xi_a^2}{b^2}}. \qquad (2.4)$$

Our definition of the rapidity parameters are *different* than those in [74].

Straightforward algebra allows us to verify that these transformations leave the interval of eq-(2.1) in classical phase space invariant. They are are fully consistent with Born's duality Relativity symmetry principle [49] $(Q,P) \rightarrow (P,-Q)$. By inspection we can see that under Born duality, the transformations in eqs-(2.2a, 2.2d) are *rotated* into each other, up to numerical b factors in order to match units. When on sets $\xi_a = 0$ in (2.2a, 2.2d) one recovers automatically the standard Lorentz transformations for the X, T and E, P variables *separately*, leaving invariant the intervals $dT^2 - dX^2 = (d\tau)^2$ and $(dE^2 - dP^2)/b^2$ separately.

When one sets $\xi_v = 0$ we obtain the transformations rules of the events in Phase space, from one reference-frame into another *uniformly*-accelerated frame of reference, $a = \text{const}$, whose acceleration-rapidity parameter is in this particular case:

$$\xi \equiv \frac{\xi_a}{b}, \quad \tanh(\xi) = \frac{ma}{m_P A_{max}}. \qquad (2.5)$$

The transformations for pure acceleration-boosts in Phase Space are:

$$T' = T\cosh\xi + \frac{P}{b}\sinh\xi, \qquad (2.6a)$$

$$E' = E\cosh\xi - bX\sinh\xi, \qquad (2.6b)$$

$$X' = X\cosh\xi - \frac{E}{b}\sinh\xi, \qquad (2.6c)$$

$$P' = P\cosh\xi + bT\sinh\xi. \qquad (2.6d)$$

It is straightforward to verify that the transformations (2.6a, 2.6c) leave invariant the fully phase space interval (2.1) but *does not* leave invariant the proper time interval $(d\tau)^2 = dT^2 - dX^2$. Only the *combination*:

$$(d\omega)^2 = (d\tau)^2\left(1 - \frac{m^2 g^2}{m_P^2 A^2_{max}}\right) \qquad (2.7a)$$

is truly left invariant under pure acceleration-boosts in Phase Space. Once again, can verify as well that these transformations satisfy Born's duality symmetry principle:

$$(T,X) \rightarrow (E,P), \; (E,P) \rightarrow (-T,-X) \qquad (2.7b)$$

and $b \rightarrow \frac{1}{b}$. The latter Born duality transformation is nothing but a manifestation of the large/small tension duality principle reminiscent of the T-duality symmetry in string theory; i. e. namely, a small/large radius duality, a winding modes/Kaluza-Klein modes duality symmetry in string compactifications and the Ultraviolet/Infrared entanglement in Noncommutative Field Theories. Hence, Born's duality principle in exchanging coordinates for momenta could be the underlying physical reason behind T-duality in string theory.

The composition of two successive pure acceleration-boosts is another pure acceleration-boost with acceleration rapidity given by $\xi'' = \xi + \xi'$. The addition of *proper* forces (accelerations) follows the usual relativistic composition rule:

$$\tanh\xi'' = \tanh(\xi+\xi') = \\ = \frac{\tanh\xi+\tanh\xi'}{1+\tanh\xi\tanh\xi'} \Rightarrow \frac{ma''}{m_P A} = \frac{\frac{ma}{m_P A}+\frac{ma'}{m_P A}}{1+\frac{m^2 aa'}{m_P^2 A^2}} \tag{2.8}$$

and in this fashion the upper limiting *proper* acceleration is never *surpassed* like it happens with the ordinary Special Relativistic addition of velocities.

The group properties of the full combination of velocity and acceleration boosts eqs-(2.2a, 2.2d) in Phase Space requires much more algebra [68]. A careful study reveals that the composition *rule* of two successive full transformations is given by $\xi''=\xi+\xi'$ and the transformation laws are *preserved* if, and only if, the $\xi;\xi';\xi''\ldots$ parameters obeyed the suitable relations:

$$\frac{\xi_a}{\xi} = \frac{\xi'_a}{\xi'} = \frac{\xi''_a}{\xi''} = \frac{\xi''_a}{\xi+\xi'}\,, \tag{2.9a}$$

$$\frac{\xi_v}{\xi} = \frac{\xi'_v}{\xi'} = \frac{\xi''_v}{\xi''} = \frac{\xi''_v}{\xi+\xi'}\,. \tag{2.9b}$$

Finally we arrive at the composition law for the effective, velocity and acceleration boosts parameters ξ''; ξ''_v; ξ''_a respectively:

$$\xi''_v = \xi_v + \xi'_v\,, \tag{2.10a}$$

$$\xi''_a = \xi_a + \xi'_a\,, \tag{2.10b}$$

$$\xi'' = \xi + \xi'. \tag{2.10c}$$

The above relations among the parameters are required in order to prove the $U(1,1)$ *group* composition law of the transformations in order to have a truly Maximal-Acceleration Phase Space Relativity theory resulting from a Phase-Space change of coordinates in the cotangent bundle of spacetime.

2.1 Planck-scale Areas are invariant under acceleration boosts

Having displayed explicitly the Group transformations rules of the coordinates in Phase space we will show why *infinite* acceleration-boosts (which is *not* the same as infinite proper acceleration) preserve Planck-scale *Areas* [68] as a result of the fact that $b=(1/L_P^2)$ equals the *maximal* invariant force, or string tension, if the units of $\hbar=c=1$ are used.

At Planck-scale L_P intervals/increments in one reference frame we have by definition (in units of $\hbar=c=1$): $\Delta X = \Delta T = L_P$ and $\Delta E = \Delta P = \frac{1}{L_P}$ where $b\equiv\frac{1}{L_P^2}$ is the maximal tension. From eqs-(2.6a, 2.6d) we get for the transformation rules of the finite intervals ΔX, ΔT, ΔE, ΔP, from one reference frame into another frame, in the *infinite* acceleration-boost limit $\xi\to\infty$,

$$\Delta T' = L_P(\cosh\xi+\sinh\xi)\to\infty$$

$$\Delta E' = \frac{1}{L_P}(\cosh\xi-\sinh\xi)\to 0 \tag{2.11b}$$

by a simple use of L'Hôpital's rule or by noticing that both $\cosh\xi;\sinh\xi$ functions approach infinity at the same rate

$$\Delta X' = L_P(\cosh\xi-\sinh\xi)\to 0\,, \tag{2.11c}$$

$$\Delta P' = \frac{1}{L_P}(\cosh\xi+\sinh\xi)\to\infty\,, \tag{2.11d}$$

where the discrete displacements of two events in Phase Space are defined: $\Delta X = X_2 - X_1 = L_P$, $\Delta E = E_2 - E_1 = \frac{1}{L_P}$, $\Delta T = T_2 - T_1 = L_P$ and $\Delta P = P_2 - P_1 = \frac{1}{L_P}$.

Due to the identity:

$$(\cosh\xi+\sinh\xi)(\cosh\xi-\sinh\xi) = \\ = \cosh^2\xi - \sinh^2\xi = 1 \tag{2.12}$$

one can see from eqs-(2.11a, 2.11d) that the Planck-scale *Areas* are truly *invariant* under *infinite* acceleration-boosts $\xi=\infty$:

$$\Delta X'\Delta P' = 0\times\infty = \\ = \Delta X\Delta P(\cosh^2\xi-\sinh^2\xi) = \Delta X\Delta P = \frac{L_P}{L_P} = 1\,, \tag{2.13a}$$

$$\Delta T'\Delta E' = \infty\times 0 = \\ = \Delta T\Delta E\,(\cosh^2\xi-\sinh^2\xi) = \Delta T\Delta E = \frac{L_P}{L_P} = 1\,, \tag{2.13b}$$

$$\Delta X'\Delta T' = 0\times\infty = \\ = \Delta X\Delta T\,(\cosh^2\xi-\sinh^2\xi) = \Delta X\Delta T = (L_P)^2\,, \tag{2.13c}$$

$$\Delta P'\Delta E' = \infty\times 0 = \\ = \Delta P\Delta E\,(\cosh^2\xi-\sinh^2\xi) = \Delta P\Delta E = \frac{1}{L_P^2}\,. \tag{2.13d}$$

It is important to emphasize that the invariance property of the minimal Planck-scale *Areas* (maximal Tension) is *not* an exclusive property of *infinite* acceleration boosts $\xi=\infty$, but, as a result of the identity $\cosh^2\xi-\sinh^2\xi=1$, for all values of ξ, the minimal Planck-scale *Areas* are *always* invariant under *any* acceleration-boosts transformations. Meaning physically, in units of $\hbar=c=1$, that the Maximal Tension (or maximal Force) $b=\frac{1}{L_P^2}$ is a true physical *invariant* universal quantity. Also we notice that the Phase-space areas, or cells, in units of $\hbar$, are also invariant! The pure-acceleration boosts transformations are "symplectic". It can be shown also that areas greater (smaller) than the Planck-area remain greater (smaller) than the invariant Planck-area under acceleration-boosts transformations.

The infinite acceleration-boosts are closely related to the infinite red-shift effects when light signals barely escape Black hole Horizons reaching an asymptotic observer with an infinite red shift factor. The important fact is that the Planck-scale *Areas* are truly maintained invariant under acceleration-boosts. This could reveal very important information about Black-holes Entropy and Holography.

3 Modified Newtonian Dynamics from Phase Space Relativity

3.1 The Machian Principle and Eddington-Dirac Large Numbers Coincidence

A natural action associated with the invariant interval in Phase-Space given by eq-(2.1) is:

$$S = m \int d\tau \sqrt{1 + \frac{m^2}{m_P^2 a^2} (d^2x^\mu/d\tau^2)(d^2x_\mu/d\tau^2)} \,. \quad (3.1)$$

The proper-acceleration is orthogonal to the proper-velocity and this can be easily verified by differentiating the time-like proper-velocity squared:

$$\begin{aligned} V^2 = \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = V^\mu V_\mu = 1 > 0 \Rightarrow \\ \Rightarrow \frac{dV^\mu}{d\tau} V_\mu = \frac{d^2x^\mu}{d\tau^2} V_\mu = 0 \,, \end{aligned} \quad (3.2)$$

which implies that the proper-acceleration is space-like:

$$\begin{aligned} -g^2(\tau) = \frac{d^2x^\mu}{d\tau^2}\frac{d^2x_\mu}{d\tau^2} < 0 \Rightarrow \\ \Rightarrow S = m \int d\tau \sqrt{1 - \frac{m^2 g^2}{m_P^2 a^2}} = m \int d\omega \,, \end{aligned} \quad (3.3)$$

where the analog of the Lorentz time-dilation factor in Phase-space is now given by

$$d\omega = d\tau \sqrt{1 - \frac{m^2 g^2(\tau)}{m_P^2 a^2}} \,, \quad (3.4a)$$

namely,

$$(d\omega)^2 = \Omega^2 d\tau^2 = \left[1 - \frac{m^2 g^2(\tau)}{m_P^2 a^2}\right] g_{\mu\nu} dx^\mu dx^\nu . \quad (3.4b)$$

The invariant proper interval is no longer the standard proper-time τ but is given by the quantity $\omega(\tau)$. The deep connection between the physics of maximal acceleration and Finsler geometry has been analyzed by [56]. The action is real-valued if, and only if, $m^2 g^2 < m_P^2 a^2$ in the same fashion that the action in Minkowski spacetime is real-valued if, and only if, $v^2 < c^2$. This is the physical reason why there is an upper bound in the proper-four force acting on a fundamental particle given by $(mg)_{bound} = m_P(c^2/L_P) = m_P^2$ in natural units of $\hbar = c = 1$.

The Eddington-Dirac large numbers coincidence (and an ultraviolet/infrared entanglement) can be easily implemented if one equates the upper bound on the proper-four force sustained by a fundamental particle, $(mg)_{bound} = m_P(c^2/L_P)$, with the proper-four force associated with the mass of the (observed) universe M_U, and whose *minimal* acceleration c^2/R is given in terms of an infrared-cutoff R (the Hubble horizon radius). Equating these proper-four forces gives

$$\frac{m_P c^2}{L_P} = \frac{M_U c^2}{R} \Rightarrow \frac{M_U}{m_P} = \frac{R}{L_P} \sim 10^{61} \,, \quad (3.5)$$

from this equality of proper-four forces associated with a maximal/minimal acceleration one infers $M_U \sim 10^{61}\, m_{Planck} \sim 10^{61} 10^{19} m_{proton} = 10^{80} m_{proton}$ which is indeed consistent with observations and agrees with the Eddington-Dirac number 10^{80}:

$$N = 10^{80} = (10^{40})^2 \sim \left(\frac{F_e}{F_G}\right)^2 \sim \left(\frac{R}{r_e}\right)^2 , \quad (3.6)$$

where $F_e = e^2/r^2$ is the electrostatic force between an electron and a proton; $F_G = G m_e m_{proton}/r^2$ is the corresponding gravitational force and $r_e = e^2/m_e \sim 10^{-13}$cm is the classical electron radius (in units $\hbar = c = 1$).

One may notice that the above equation (3.5) is also consistent with the Machian postulate that the rest mass of a particle is determined via the gravitational potential energy due to the other masses in the universe. In particular, by equating:

$$m_i c^2 = G m_i \sum_j \frac{m_j}{|r_i - r_j|} = \frac{G m_i M_U}{R} \Rightarrow \frac{c^2}{G} = \frac{M_U}{R} . \quad (3.7)$$

Due to the negative binding energy, the composite mass m_{12} of a system of two objects of mass m_1, m_2 is not equal to the sum $m_1 + m_2 > m_{12}$. We can now arrive at the conclusion that the *minimal* acceleration c^2/R is also the same acceleration induced on a test particle of mass m by a spherical mass distribution M_U inside a radius R. The acceleration felt by a test particle of mass m sitting at the edge of the observable Universe (at the Hubble horizon radius R) is:

$$\frac{G M_U}{R^2} = a \,. \quad (3.8)$$

From the last two equations (3.7, 3.8) one gets the same expression for the *minimal* acceleration:

$$a = a_{min} = \frac{c^2}{R} \,, \quad (3.9)$$

which is of the same order of magnitude as the anomalous acceleration of the Pioneer and Galileo spacecrafts $a \sim 10^{-8}$ cm/s^2. A very plausible physical reason behind the observed anomalous Pioneer acceleration could be due to the fact that the universe is in accelerated expansion and motion (a non-inertial frame of reference) w. r. t the vacuum. Our proposal shares some similarities with the previous work of [6]. To our knowledge, the first person who *predicted* the Pioneer anomaly in 1978 was P. LaViolette [5], from an entirely different approach based on the novel theory of sub-quantum kinetics to explain the vacuum fluctuations, two years *prior* to the Anderson et al observations [7]. Nottale has invoked the Machian principle of inertia [3] adopting a local and global inertial coordinate system at the scale of the solar system in order to explain the origins of this Pioneer-Galileo anomalous constant acceleration. The Dirac-Eddington large number coincidences from vacuum fluctuations was studied by [8].

Let us examine closer the equality between the proper-four forces

$$\frac{m_P c^2}{L_P} = \frac{M_U c^2}{R} \Rightarrow \frac{m_P}{L_P} = \frac{M_U}{R} = \frac{c^2}{G}. \tag{3.10}$$

The last term in eq-(3.10) is directly obtained after implementing the Machian principle in eq-(3.7). Thus, one concludes from eq-(3.10) that as the universe evolves in time one must have the conserved ratio of the quantities $M_U/R = c^2/G = m_P/L_P$. This interesting possibility, advocated by Dirac long ago, for the fundamental constants $\hbar$, c, G, ... to vary over cosmological time is a plausible idea with the provision that the above ratios satisfy the relations in eq-(3.10) at any given moment of cosmological time. If the fundamental constants do not vary over time then the ratio $M_U/R = c^2/G$ must refer then to the *asymptotic* values of the Hubble horizon radius $R = R_{asymptotic}$. A related approach to the idea of an impassible upper asymptotic length R has been advocated by Scale Relativity [2] and in Khare [94] where a Cosmology based on non-Archimedean geometry was proposed by recurring to p-adic numbers. For example, a Non-Archimedean number addition law of two masses m_1, m_2 does not follow the naive addition rule $m_1 + m_2$ but instead:

$$m_1 \bullet m_2 = \frac{m_1 + m_2}{1 + (m_1 m_2 / M_U^2)},$$

which is similar to the composition law of velocities in ordinary Relativity in terms of the speed of light. When the masses m_1, m_2 are much smaller than the universe mass M_U one recovers the ordinary addition law. Similar considerations follow in the Non-Archimedean composition law of lengths such that the upper length R_{asym} is never surpassed. For further references on p-adic numbers and Physics were refer to [40]. A Mersenne prime, $M_p = 2^p - 1 = prime$, for $p = prime$, p-adic hierarchy of scales in Particle physics and Cosmology has been discussed by Pitkannen and Noyes where many of the the fundamental energy scales, masses and couplings in Physics has been obtained [41], [42]. For example, the Mersenne prime $M_{127} = 2^{127} - 1 \sim 10^{38} \sim (m_{Planck}/m_{proton})^2$. The derivation of the Standard Model parameters from first principle has obtained by Smith [43] and Beck [47].

In [68] we proposed a plausible explanation of the variable fine structure constant phenomenon based on the maximal-acceleration relativity principle in phase-space by modifying the Robertson-Friedmann-Walker metric by a similar (acceleration-dependent) conformal factor as in eqs-(3.4). It led us to the conclusion that the universe could have emerged from the vacuum as a quantum bubble (or "brane-world") of Planck mass and Planck radius that expanded (w. r. t to the vacuum) at the speed of light with a *maximal* acceleration $a = c^2/L_p$. Afterwards the acceleration began to slow down as matter was being created from the vacuum, via an Unruh-Rindler-Hawking effect, from this initial maximal value c^2/L_p to the value of $c^2/R \sim 10^{-8}$cm/s^2 (of the same order of magnitude as the Pioneer anomalous acceleration). Namely, as the universe expanded, matter was being created from the vacuum via the Unruh-Rindler-Hawking effect (which must not to be confused with Hoyle's Steady State Cosmolgy) such that the observable mass of the universe enclosed within the observed Hubble horizon radius obeys (at any time) the relation $M_U \sim R$. Such latter relationship is very similar (up to a factor of 2) to the Schwarzschild black-hole horizon-radius relation $r_s = 2M$ (in units of $\hbar = c = G = 1$). As matter is being created out of the vacuum, the Hubble horizon radius grows accordingly such that $M_U/R = c^2/G$. Note that the Hubble horizon radius is one-half the Schwarzchild horizon radius $(1/2)(2GM_U/c^2) = (1/2)R_S$.

Lemaître's idea of the Universe as a "primordial atom" (like a brane-world) of Planck size has been also analyzed by [30] from a very different perspective than Born's Dual Phase Space Relativity. These authors have argued that one can have a compatible picture of the expansion of the Universe with the Eddington-Dirac large number coincidences if one invokes a variation of the fundamental constants with the cosmological evolution time as Dirac advocated long ago.

One of the most salient features of this section is that it agrees with the findings of [4] where a *geometric mean* relationship was found from first principles among the cosmological constant ρ_{vacuum}, the Planck area λ^2 and the AdS_4 throat size squared R^2 given by $(\rho_v)^{-1} = (\lambda)^2(R^2)$. Since the throat size of de Sitter space is the same as that of Anti de Sitter space, by setting the infrared scale R equal to the Hubble radius horizon observed *today* R_H and λ equal to the Planck scale one reproduces precisely the *observed* value of the vacuum energy density! [25]: $\rho \sim L_{Planck}^{-2} R_H^{-2} = L_P^{-4}(L_{Planck}/R_H)^2 \sim 10^{-122} M_{Planck}^4$.

Nottale's proposal [2] for the resolution to the cosmological constant problem is based on taking the Hubble scale R as an upper impassible scale and implementing the Scale Relativity principle so that in order to compare the vacuum energies of the Universe at the Planck scale $\rho(L_P)$ with the vacuum energy measured at the Hubble scale $\rho(R)$ one needs to include the Scale Relativistic correction factors which account for such apparent huge discrepancy: $\rho(L_P)/\rho(R) = (R/L_P)^2 \sim 10^{122}$. In contrast, the results of this work are based on Born's Dual Phase-Space Relativity principle. In the next sections we will review the dynamical consequences of the Yang's Noncommutative spacetime algebra comprised of *two* scales, the minimal Planck scale L_p (related to a minimum distance) and an upper infrared scale R related to a minimum momentum $p = \hbar/R$. Another interesting approach to dark matter, dark energy and the cosmological constant based on a vacuum condensate has been undertaken by [25].

We finalize this subsection by pointing out that the maximal/minimal angular velocity correspond to c/L_P and c/R respectively. A maximum angular velocity has important consequences in future Thomas-precession experiments [61], [73] whereas a minimal angular velocity has important consequences in galactic rotation measurements. The role of the Machian principle in constructing quantum cosmologies, models of dark energy, etc... has been studied in [52] and its relationship to modified Newtonian dynamics and fractals by [54], [3].

3.2 Modified Newtonian Dynamics from Phase-Space Relativity

Having displayed the cosmological features behind the proper-four forces equality (3.10) that relates the maximal/ minimal acceleration in terms of the minimal/large scales and which is compatible with Eddington-Dirac's large number coincidences we shall derive next the *modified* Newtonian dynamics of a test particle which emerges from the Born's Dual Phase Space Relativity principle.

The modified Schwarzschild metric is defined in terms of the non-covariant acceleration as:

$$(d\omega)^2 = \Omega^2 (d\tau)^2 = \left[1 + \frac{m^2 g_{\mu\nu}(d^2x^\mu/d\tau^2)(d^2x^\nu/d\tau^2)}{m_P^2 a^2}\right] g_{\mu\nu} dx^\mu dx^\nu \,,$$

$$-g^2(\tau) \equiv g_{\mu\nu}(d^2x^\mu/d\tau^2)(d^2x^\nu/d\tau^2) < 0\,. \qquad (3.11a)$$

A covariant acceleration in curved space-times is given by:

$$\frac{Dv^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\,.$$

A particle in free fall follows a geodesic with *zero* covariant acceleration. Hence, we shall use the non-covariant acceleration in order to compute the effects of the maximal acceleration of a test particle in Schwarzschild spacetimes.

The components of the non-covariant four-acceleration $d^2x^\mu/d\tau^2$ of a test particle of mass m moving in a Schwarzschild spacetime background can be obtained in a straightforward fashion after using the on-shell condition $g_{\mu\nu}P^\mu P^\nu = m^2$ in spherical coordinates (by solving the relativistic Hamilton-Jacobi equations). The explict components of the (space-like) proper-four acceleration can be found in [22], [36] in terms of two integration constants, the energy E and angular momentum L. The latter components yields the final expression for the conformal factor Ω^2 in the case of pure radial motion [22]:

$$\begin{aligned}\Omega^2(m,a,M,E,r) &= \\ &= 1 - \left(\frac{m}{m_P}\right)^2\left(\frac{1}{a^2}\right)\Big\{(1-2M/r)^{-1}\left(\frac{M}{r^2}\right)^2 - \\ &- \left[4M^2(E/m)^2 r^{-4}(1-2M/r)^{-3}\right]\times \\ &\times\left[(E/m)^2-(1-2M/r)\right]\Big\}\,.\end{aligned} \qquad (3.12)$$

In the Newtonian limit, to a first order approximation, we can set $1-2M/r \sim 1$ in eq-(3.12) since we shall be concentrating in distances larger than the Schwarzschild radius $r > r_s = 2M$, the conformal factor Ω^2 in eq-(3.12) simplifies:

$$\begin{aligned}\Omega^2 \sim 1 - \left(\frac{m}{m_P}\right)^2\left(\frac{1}{a^2}\right)\Big\{\left(\frac{M}{r^2}\right)^2 - \\ - \left[4M^2(E/m)^2r^{-4}\right]\left[(E/m)^2-1\right]\Big\},\end{aligned} \qquad (3.13)$$

the modified Schwarzschild metric component $g'_{00} = \Omega^2 g_{00} = \Omega^2(1-2M/r) = 1 + 2\mathcal{U}'$ yields the *modified* gravitational potential $\mathcal{U}'$ in the *weak* field approximation

$$\begin{aligned}g'_{00} &= 1 + 2\mathcal{U}' \sim \\ &\sim 1 - \frac{2M}{r} - \left(\frac{m}{m_P}\right)^2\left(\frac{1}{a^2}\right)\left(\frac{2M}{r^2}\right)^2 F(E/m)\end{aligned} \qquad (3.14)$$

with

$$F(E/m) = \left(\frac{E}{m}\right)^2 - \left(\frac{E}{m}\right)^4 + \frac{1}{4}\,, \qquad (3.15)$$

where $F(E/m) > 0$ in the Newtonian limit $E < m$. The modified radial acceleration which encodes the modified Newtonian dynamics and which violates the equivalence principle (since the acceleration now depends on the mass of the test particle m) is

$$\begin{aligned}a' = -\frac{\partial\mathcal{U}'}{\partial r} = -\frac{M}{r^2}\Bigg[1 + 8F\left(\frac{E}{m}\right)\left(\frac{m}{m_P}\right)^2 \times \\ \times\left(\frac{M}{m_P}\right)\frac{1}{m_P^3 r^3}\Bigg] + O(r^{-6})\,,\end{aligned} \qquad (3.16)$$

this result is valid for distances $r \gg r_s = 2M$. We have set the maximal acceleration $a = \frac{c^2}{L_P} = m_P$ in units of $\hbar = c = G = 1$. This explains the presence of the m_P factors in the denominators. The first term in eq-(3.16) is the standard Newtonian gravitational acceleration $-M/r^2$ and the second terms are the leading corrections of order $1/r^5$. The higher order corrections $O(r^{-6})$ appear when we do not set $1-2M/r \sim 1$ in the expression for the conformal factor Ω^2 and when we include the extra term in the product of Ω^2 with $g_{00} = (1-2M/r)$.

The conformal factor Ω^2 when $L \neq 0$ (rotational degrees of freedom are switched on) such that the test particle moves in the radial and transverse (angular) directions has been given in [22]:

$$\begin{aligned}\Omega^2 = 1 - \frac{m^2}{m_P^2 a^2}\Bigg\{\frac{1}{1-2M/r} \times \\ \times \left[-\frac{3ML^2}{m^2r^4} + \frac{L^2}{m^2r^3} - \frac{M}{r^2}\right]^2\Bigg\} + \\ + \frac{m^2}{m_P^2 a^2}\left[-\frac{4L^2}{m^2r^4} + \frac{4E^2M^2}{m^2r^4(1-2M/r)^3}\right] \times \\ \times \left[\frac{E^2}{m^2} - (1-2M/r)\left(1+\frac{L^2}{m^2r^2}\right)\right].\end{aligned} \tag{3.17}$$

Following the same weak field approximation procedure $g'_{00} = \Omega^2(E, L, m)g_{00} = 1 + 2\mathcal{U}'$ yields the modified gravitational potential $\mathcal{U}'$ and modified Newtonian dynamics $a' = -\partial_r \mathcal{U}'$ that leads once again to a violation of the equivalence principle due to the fact that the acceleration depends on the values of the masses of the test particle.

4 Modified Newtonian Dynamics resulting from Yang's Noncommutative Spacetime Algebra

We end this work with some relevant remarks about the impact of Yang's Noncommutative spacetime algebra on modified Newtonian dynamics. Such algebra involves *two* length scales, the minimal Planck scale $L_P = \lambda$ and an upper infrared cutoff scale $\mathcal{R}$.

Recently in [134] an isomorphism between Yang's Noncommutative space-time algebra (involving *two* length scales) [136] and the *holographic area coordinates* algebra of C-spaces (Clifford spaces) was constructed via an AdS_5 spacetime (embedded in $6D$) which is instrumental in explaining the origins of an extra (infrared) scale $\mathcal{R}$ in conjunction to the (ultraviolet) Planck scale λ characteristic of C-spaces. Yang's Noncommutative space-time algebra allowed Tanaka [137] to explain the origins behind the *discrete* nature of the spectrum for the *spatial* coordinates and *spatial* momenta which yields a *minimum* length-scale λ (ultraviolet cutoff) and a minimum momentum $p = \hbar/\mathcal{R}$ (maximal length $\mathcal{R}$, infrared cutoff).

Related to the issue of area-quantization, the norm-squared $\mathbf{A}^2$ of the holographic Area operator $X_{AB}X^{AB}$ in Clifford-spaces has a correspondence with the quadratic Casimir operator $\lambda^4 \Sigma_{AB}\Sigma^{AB}$ of the conformal algebra $SO(4, 2)$ ($SO(5, 1)$ in the Euclideanized AdS_5 case). This holographic area-Casimir relationship does not differ much from the area-spin relation in Loop Quantum Gravity $\mathbf{A}^2 \sim \lambda^4 \sum j_i(j_i + 1)$ in terms of the $SU(2)$ Casimir J^2 with eigenvalues $j(j + 1)$, where the sum is taken over the spin network sites [111] and the minimal Planck scale emerges from a regularization procedure.

The Yang's algebra can be written in terms of the $6D$ angular momentum operators and a $6D$ pseudo-Euclidean metric η^{MN}:

$$\hat{M}^{\mu\nu} = \hbar\Sigma^{\mu\nu}, \quad \hat{M}^{56} = \hbar\Sigma^{56}, \tag{4.1}$$

$$\lambda\Sigma^{\mu5} = \hat{x}^\mu, \quad \frac{\hbar}{\mathcal{R}}\Sigma^{\mu6} = \hat{p}^\mu, \tag{4.2}$$

$$\mathcal{N} = \frac{\lambda}{\mathcal{R}}\Sigma^{56}, \tag{4.3}$$

as follows:

$$[\hat{p}^\mu, \mathcal{N}] = -i\eta^{66}\frac{\hbar}{R^2}\,\hat{x}^\mu, \tag{4.4}$$

$$[\hat{x}^\mu, \mathcal{N}] = i\eta^{55}\frac{L_P^2}{\hbar}\,\hat{p}^\mu, \tag{4.5}$$

$$[\hat{x}^\mu, \hat{x}^\nu] = -i\eta^{55}L_P^2\,\Sigma^{\mu\nu}, \tag{4.6}$$

$$[\hat{p}^\mu, \hat{p}^\nu] = -i\eta^{66}\frac{\hbar^2}{R^2}\,\Sigma^{\mu\nu}, \tag{4.7}$$

$$[\hat{x}^\mu, \hat{p}^\mu] = i\hbar\eta^{\mu\nu}\,\mathcal{N}, \tag{4.8}$$

$$[\hat{x}^\mu, \Sigma^{\nu\rho}] = \eta^{\mu\rho}x^\nu - \eta^{\mu\nu}x^\rho, \tag{4.9}$$

$$[\hat{p}^\mu, \Sigma^{\nu\rho}] = \eta^{\mu\rho}p^\nu - \eta^{\mu\nu}p^\rho, \tag{4.10}$$

The dynamical consequences of the Yang's Noncommutative spacetime algebra can be derived from the quantum/classical correspondence:

$$\frac{1}{i\hbar}[\hat{A}, \hat{B}] \leftrightarrow \{A, B\}_{PB}, \tag{4.11}$$

i. e. commutators correspond to Poisson brackets. More precisely, to Moyal brackets in Phase Space. In the classical limit $\hbar \to 0$ Moyal brackets reduce to Poisson brackets. Since the coordinates and momenta are no longer commuting variables the classical Newtonian dynamics is going to be modified since the symplectic two-form $\omega^{\mu\nu}$ in Phase Space will have additional non-vanishing elements stemming from these non-commuting coordinates and momenta.

In particular, the modified brackets read now:

$$\begin{aligned}\{\{A(x,p), B(x,p)\}\} = \partial_\mu A\omega^{\mu\nu}\partial_\nu B = \\ = \{A(x,p), B(x,p)\}_{PB}\{x^\mu, p^\nu\} + \\ + \frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial x^\nu}\{x^\mu, x^\nu\} + \frac{\partial A}{\partial p^\mu}\frac{\partial B}{\partial p^\nu}\{p^\mu, p^\nu\}.\end{aligned} \tag{4.12}$$

If the coordinates and momenta were commuting variables the modified bracket will reduce to the first term only:

$$\begin{aligned}\{\{A(x,p), B(x,p)\}\} = \\ = \{A(x,p), B(x,p)\}_{PB}\{x^\mu, p^\nu\} = \\ = \left[\frac{\partial A}{\partial x^\mu}\frac{\partial B}{\partial p^\nu} - \frac{\partial A}{\partial p^\mu}\frac{\partial B}{\partial x^\nu}\right]\eta^{\mu\nu}\mathcal{N}.\end{aligned} \tag{4.13}$$

The ordinary Heisenberg (canonical) algebra is recovered when $\mathcal{N} \to 1$ in eq-(4.13).

In the nonrelativistic limit, the modified dynamical equations are:

$$\frac{dx^i}{dt} = \{\{x^i, H\}\} = \frac{\partial H}{\partial p^j}\{x^i, p^j\} + \frac{\partial H}{\partial x^j}\{x^i, x^j\}, \quad (4.14)$$

$$\frac{dp^i}{dt} = \{\{p^i, H\}\} = -\frac{\partial H}{\partial x^j}\{x^i, p^j\} + \frac{\partial H}{\partial p^j}\{p^i, p^j\}. \quad (4.15)$$

The non-relativistic Hamiltonian for a central potential $V(r)$ is:

$$H = \frac{p_i p^i}{2m} + V(r), \quad r = \Big[\sum_i x_i x^i\Big]^{1/2}. \quad (4.16)$$

Defining the magnitude of the central force by $F = -\frac{\partial V}{\partial r}$ and using $\frac{\partial r}{\partial x^i} = \frac{x_i}{r}$ one has the modified dynamical equations of motion (4.14, 4.15):

$$\frac{dx^i}{dt} = \{\{x^i, H\}\} = \frac{p_j}{m}\delta^{ij} - F\frac{x_j}{r}L_P^2\Sigma^{ij}, \quad (4.16a)$$

$$\frac{dp^i}{dt} = \{\{p^i, H\}\} = F\frac{x_j}{r}\delta^{ij} + \frac{p_j}{m}\frac{\Sigma^{ij}}{R^2}. \quad (4.16b)$$

The angular momentum two-vector Σ^{ij} can be written as the *dual* of a vector $\vec{J}$ as follows $\Sigma^{ij} = \epsilon^{ijk}J_k$ so that:

$$\frac{dx^i}{dt} = \{\{x^i, H\}\} = \frac{p^i}{m} - L_P^2 F\frac{x_j}{r}\epsilon^{ijk}J_k, \quad (4.17a)$$

$$\frac{dp^i}{dt} = \{\{p^i, H\}\} = F\frac{x^i}{r} + \frac{p_j}{m}\frac{\epsilon^{ijk}J_k}{R^2}. \quad (4.17b)$$

For planar motion (central forces) the cross-product of $\vec{J}$ with $\vec{p}$ and $\vec{x}$ is not zero since $\vec{J}$ points in the perpendicular direction to the plane. Thus, one will have nontrivial corrections to the ordinary Newtonian equations of motion induced from Yang's Noncommutative spacetime algebra in the non-relativistic limit. When $\vec{J} = 0$, pure radial motion, there are no corrections. This is not the case when we studied the modified Newtonian dynamics in the previous section of the modified Schwarzschild field due to the maximal-acceleration relativistic effects. Therefore, the two routes to obtain modifications of Newtonian dynamics are very different.

Concluding, eqs-(4.16, 4.17) determine the *modified* Newtonian dynamics of a test particle under the influence of a central potential explicitly in terms of the two L_P, R minimal/maximal scales. When $L_P \to 0$ and $R \to \infty$ one recovers the ordinary Newtonian dynamics $v^i = (p^i/m)$ and $F(x^i/r) = m(dv^i/dt)$. The unit vector in the radial direction has for components $\hat{r} = (\vec{r}/r) = (x^1/r, x^2/r, x^3/r)$.

It is warranted to study the full relativistic dynamics as well, in particular the *modified* relativistic dynamics of the de-Sitter rigid top [135] due to the effects of Yang's Noncommutative spacetime algebra with a lower and an upper scale. The de Sitter rigid Top can be generalized further to Clifford spaces since a Clifford-polyparticle has more degrees of freedom than a relativistic top in ordinary spacetimes [46] and, naturally, to study the *modified* Nambu-Poisson dynamics of p-branes [49] as well. A different physical approach to the theory of large distance physics based on certain two-dim nonlinear sigma models has been advanced by Friedan [51].

An Extended Relativity theory with both an upper and lower scale can be formulated in the Clifford extension of Phase Spaces along similar lines as [1], [68] by adding the Clifford-valued polymomentum degrees of freedom to the Clifford-valued holographic coordinates. The Planck scale L_P and the minimum momentum $(\hbar/R)$ are introduced to match the dimensions in the Clifford-Phase Space interval in D-dimensions as follows:

$$\begin{aligned} d\Sigma^2 &= <dX^\dagger dX> + \frac{1}{\mathcal{F}^2}<dP^\dagger dP> = \\ &= \left(\frac{d\sigma}{L_P^{D-1}}\right)^2 + dx_\mu dx^\mu + \frac{dx_{\mu\nu}dx^{\mu\nu}}{L_P^2} + \\ &+ \frac{dx_{\mu\nu\rho}dx^{\mu\nu\rho}}{L_P^4} + \ldots + \frac{1}{\mathcal{F}^2}\left[\left(\frac{d\tilde{\sigma}}{(\hbar/R)^{D-1}}\right)^2 + \right. \\ &\left. + dp_\mu dp^\mu + \frac{dp_{\mu\nu}dp^{\mu\nu}}{(\hbar/R)^2} + \frac{dp_{\mu\nu\rho}dp^{\mu\nu\rho}}{(\hbar/R)^4} + \ldots\right]. \end{aligned} \quad (4.18)$$

All the terms in eq-(4.18) have dimensions of length^2 and the maximal force is:

$$\mathcal{F} = \frac{m_P c^2}{L_P} = \frac{M_U c^2}{R} = \frac{c^4}{G}. \quad (4.19)$$

The relevance of studying this extended Relativity in a Clifford-extended Phase Space is that it is the proper arena to construct a Quantum Cosmology compatible with Non-Archimedean Geometry, Yang's Noncommutative spacetime algebra [136] and Scale Relativity [2] with an upper and lower limiting scales, simultaneously. This clearly deserves further investigation.

Acknowledgments

We are indebted to M. Bowers for assistance.

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