On Nonlinear Quantum Mechanics, Brownian Motion, Weyl Geometry and Fisher Information

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A new nonlinear Schrödinger equation is obtained explicitly from the (fractal) Brownian motion of a massive particle with a complex-valued diffusion constant. Real-valued energy plane-wave solutions and solitons exist in the free particle case. One remarkable feature of this nonlinear Schrödinger equation based on a (fractal) Brownian motion model, over all the other nonlinear QM models, is that the quantum-mechanical energy functional coincides precisely with the field theory one. We finalize by showing why a *complex* momentum is essential to fully understand the physical implications of Weyl's geometry in QM, along with the interplay between Bohm's Quantum potential and Fisher Information which has been overlooked by several authors in the past.

1 Introduction

Over the years there has been a considerable debate as to whether linear QM can fully describe Quantum Chaos. Despite that the quantum counterparts of classical chaotic systems have been studied via the techniques of linear QM, it is our opinion that Quantum Chaos is truly a new paradigm in physics which is associated with non-unitary and nonlinear QM processes based on non-Hermitian operators (implementing time symmetry breaking). This Quantum Chaotic behavior should be linked more directly to the Nonlinear Schrödinger equation without any reference to the nonlinear behavior of the classical limit. For this reason, we will analyze in detail the fractal geometrical features underlying our Nonlinear Schrödinger equation obtained in [6].

Nonlinear QM has a practical importance in different fields, like condensed matter, quantum optics and atomic and molecular physics; even quantum gravity may involve nonlinear QM. Another important example is in the modern field of quantum computing. If quantum states exhibit small nonlinearities during their temporal evolution, then quantum computers can be used to solve NP-complete (non polynomial) and #P problems in polynomial time. Abrams and Lloyd [19] proposed logical gates based on non linear Schrödinger equations and suggested that a further step in quantum computing consists in finding physical systems whose evolution is amenable to be described by a NLSE.

On other hand, we consider that Nottale and Ord's formulation of quantum mechanics [1], [2] from first principles based on the combination of scale relativity and fractal spacetime is a very promising field of future research. In this work we extend Nottale and Ord's ideas to derive the nonlinear Schrödinger equation. This could shed some light on the physical systems which could be appropriately described by the nonlinear Schrödinger equation derived in what follows.

The contents of this work are the following: In section 2 we derive the nonlinear Schrödinger equation by extending Nottale-Ord's approach to the case of a fractal Brownian motion with a complex diffusion constant. We present a thorough analysis of such nonlinear Schrödinger equation and show why it cannot linearized by a naive complex scaling of the wavefunction $\psi \to \psi^\lambda$.

Afterwards we will describe the explicit interplay between Fisher Information, Weyl geometry and the Bohm's potential by introducing an action based on a *complex* momentum. The connection between Fisher Information and Bohm's potential has been studied by several authors [24], however the importance of introducing a *complex* momentum $P_k = p_k + iA_k$ (where A_k is the Weyl gauge field of dilatations) in order to fully understand the physical implications of Weyl's geometry in QM, along with the interplay between Bohm's quantum potential and Fisher Information, has been overlooked by several authors in the past [24], [25]. For this reason we shall review in section 3 the relationship between Bohm's Quantum Potential and the Weyl curvature scalar of the Statistical ensemble of particle-paths (an Abelian fluid) associated to a single particle that was initially developed by [22]. A Weyl geometric formulation of the Dirac equation and the nonlinear Klein-Gordon wave equation was provided by one of us [23]. In the final section 4, we summarize our conclusions and include some additional comments.

2 Nonlinear QM as a fractal Brownian motion with a complex diffusion constant

We will be following very closely Nottale's derivation of the ordinary Scrödinger equation [1]. Recently Nottale and

Celerier [1] following similar methods were able to derive the Dirac equation using bi-quaternions and after breaking the parity symmetry $dx^\mu \leftrightarrow -dx^\mu$, see references for details. Also see the Ord's paper [2] and the Adlers's book on quaternionic QM [16]. For simplicity the one-particle case is investigated, but the derivation can be extended to many-particle systems. In this approach particles do not follow smooth trajectories but fractal ones, that can be described by a continuous but non-differentiable fractal function $\vec{r}(t)$. The time variable is divided into infinitesimal intervals dt which can be taken as a given scale of the resolution.

Then, following the definitions given by Nelson in his stochastic QM approach (Lemos in [12] p. 615; see also [13, 14]), Nottale define mean backward an forward derivatives

$$\frac{d_\pm \vec{r}(t)}{dt} = \lim_{\Delta t \to \pm 0} \left\langle \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \right\rangle, \tag{1}$$

from which the forward and backward mean velocities are obtained,

$$\frac{d_\pm \vec{r}(t)}{dt} = \vec{b}_\pm\,. \tag{2}$$

For his deduction of Schrödinger equation from this fractal space-time classical mechanics, Nottale starts by defining the complex-time derivative operator

$$\frac{\delta}{dt} = \frac{1}{2}\left(\frac{d_+}{dt} + \frac{d_-}{dt}\right) - i\,\frac{1}{2}\left(\frac{d_+}{dt} - \frac{d_-}{dt}\right), \tag{3}$$

which after some straightforward definitions and transformations takes the following form,

$$\frac{\delta}{dt} = \frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla} - iD\nabla^2, \tag{4}$$

D is a real-valued diffusion constant to be related to the Planck constant.

The D comes from considering that the scale dependent part of the velocity is a Gaussian stochastic variable with zero mean, (see de la Peña at [12] p. 428)

$$\langle d\xi_{\pm i}\, d\xi_{\pm j}\rangle = \pm 2D\delta_{ij}dt\,. \tag{5}$$

In other words, the fractal part of the velocity $\vec{\xi}$, proportional to the $\vec{\zeta}$, amount to a Wiener process when the fractal dimension is 2.

Afterwards, Nottale defines a set of complex quantities which are generalization of well known classical quantities (Lagrange action, velocity, momentum, etc), in order to be coherent with the introduction of the complex-time derivative operator. The complex time dependent wave function ψ is expressed in terms of a Lagrange action S by $\psi = e^{iS/(2mD)}$. S is a complex-valued action but D is real-valued. The velocity is related to the momentum, which can be expressed as the gradient of S, $\vec{p} = \vec{\nabla}S$. Then the following known relation is found,

$$\vec{V} = -2iD\vec{\nabla}\ln\psi\,. \tag{6}$$

The Schrödinger equation is obtained from the Newton's equation (force = mass times acceleration) by using the expression of $\vec{V}$ in terms of the wave function ψ,

$$-\vec{\nabla}U = m\frac{\delta}{dt}\vec{V} = -2imD\frac{\delta}{dt}\vec{\nabla}\ln\psi\,. \tag{7}$$

Replacing the complex-time derivation (4) in the Newton's equation gives us

$$-\vec{\nabla}U = -2im\left(D\frac{\partial}{\partial t}\vec{\nabla}\ln\psi\right) - 2D\vec{\nabla}\left(D\frac{\nabla^2\psi}{\psi}\right). \tag{8}$$

Simple identities involving the $\vec{\nabla}$ operator were used by Nottale. Integrating this equation with respect to the position variables finally yields

$$D^2\nabla^2\psi + iD\frac{\partial\psi}{\partial t} - \frac{U}{2m}\psi = 0\,, \tag{9}$$

up to an arbitrary phase factor which may set to zero. Now replacing D by $\hbar/(2m)$, we get the Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi = U\psi\,. \tag{10}$$

The Hamiltonian operator is Hermitian, this equation is linear and clearly is homogeneous of degree one under the substitution $\psi \to \lambda\psi$.

Having reviewed Nottale's work [1] we can generalize it by relaxing the assumption that the diffusion constant is real; we will be working with a complex-valued diffusion constant; i. e. with a complex-valued $\hbar$. This is our new contribution. The reader may be immediately biased against such approach because the Hamiltonian ceases to be Hermitian and the energy becomes complex-valued. However this is not always the case. We will explicitly find plane wave solutions and soliton solutions to the nonlinear and non-Hermitian wave equations with real energies and momenta. For a detailed discussion on complex-valued spectral representations in the formulation of quantum chaos and time-symmetry breaking see [10]. Nottale's derivation of the Schrödinger equation in the previous section required a complex-valued action S stemming from the complex-valued velocities due to the breakdown of symmetry between the forwards and backwards velocities in the fractal zigzagging. If the action S was complex then it is not farfetched to have a complex diffusion constant and consequently a complex-valued $\hbar$ (with same units as the complex-valued action).

Complex energy is not alien in ordinary linear QM. They appear in optical potentials (complex) usually invoked to model the absorption in scattering processes [8] and decay of unstable particles. Complex potentials have also been used to describe decoherence. The accepted way to describe resonant states in atomic and molecular physics is based on the complex scaling approach, which in a natural way deals with complex energies [17]. Before, Nottale wrote,

$$\langle d\zeta_\pm\, d\zeta_\pm\rangle = \pm 2Ddt\,, \tag{11}$$

with D and $2mD = \hbar$ real. Now we set

$$\langle d\zeta_\pm \, d\zeta_\pm \rangle = \pm (D + D^*)\, dt\,, \tag{12}$$

with D and $2mD = \hbar = \alpha + i\beta$ complex. The complex-time derivative operator becomes now

$$\frac{\delta}{dt} = \frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla} - \frac{i}{2}(D + D^*)\nabla^2. \tag{13}$$

In the real case $D = D^*$. It reduces to the complex-time-derivative operator described previously by Nottale. Writing again the ψ in terms of the complex action S,

$$\psi = e^{iS/(2mD)} = e^{iS/\hbar}, \tag{14}$$

where S, D and $\hbar$ are complex-valued, the complex velocity is obtained from the complex momentum $\vec{p} = \vec{\nabla} S$ as

$$\vec{V} = -2iD\vec{\nabla}\ln\psi\,. \tag{15}$$

The NLSE (non-linear Schröedinger equation) is obtained after we use the generalized Newton's equation (force = mass times acceleration) in terms of the ψ variable,

$$-\vec{\nabla} U = m\,\frac{\delta}{dt}\,\vec{V} = -2imD\frac{\delta}{dt}\vec{\nabla}\ln\psi\,. \tag{16}$$

Replacing the complex-time derivation (13) in the generalized Newton's equation gives us

$$\vec{\nabla} U = 2im\left[D\frac{\partial}{\partial t}\vec{\nabla}\ln\psi - 2iD^2(\vec{\nabla}\ln\psi\cdot\vec{\nabla})\times \times(\vec{\nabla}\ln\psi) - \frac{i}{2}(D+D^*)D\nabla^2(\vec{\nabla}\ln\psi)\right]. \tag{17}$$

Now, using the next three identities: (i) $\vec{\nabla}\nabla^2 = \nabla^2\vec{\nabla}$; (ii) $2(\vec{\nabla}\ln\psi\cdot\vec{\nabla})(\vec{\nabla}\ln\psi) = \vec{\nabla}(\vec{\nabla}\ln\psi)^2$; and (iii) $\nabla^2\ln\psi = \nabla^2\psi/\psi - (\vec{\nabla}\ln\psi)^2$ allows us to integrate such equation above yielding, after some straightforward algebra, the NLSE

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\alpha}{\hbar}\nabla^2\psi + U\psi - i\,\frac{\hbar^2}{2m}\frac{\beta}{\hbar}\left(\vec{\nabla}\ln\psi\right)^2\psi\,. \tag{18}$$

Note the crucial minus sign in front of the kinematic pressure term and that $\hbar = \alpha + i\beta = 2mD$ is complex. When $\beta = 0$ we recover the linear Schrödinger equation.

The nonlinear potential is now complex-valued in general. Defining

$$W = W(\psi) = -\frac{\hbar^2}{2m}\frac{\beta}{\hbar}\left(\vec{\nabla}\ln\psi\right)^2, \tag{19}$$

and U the ordinary potential, we rewrite the NLSE as

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\alpha}{\hbar}\nabla^2 + U + iW\right)\psi\,. \tag{20}$$

This is the fundamental nonlinear wave equation of this work. It has the form of the ordinary Schrödinger equation with the complex potential $U + iW$ and the complex $\hbar$. The Hamiltonian is no longer Hermitian and the potential $V = U + iW(\psi)$ itself depends on ψ. Nevertheless one could have meaningful physical solutions with real valued energies and momenta, like the plane-wave and soliton solutions studied in the next section. Here are some important remarks.

• Notice that the NLSE above *cannot* be obtained by a *naive* scaling of the wavefunction

$$\psi = e^{iS/\hbar_0} \to \psi' = e^{iS/\hbar} = e^{(iS/\hbar_0)(\hbar_0/\hbar)} = = \psi^\lambda = \psi^{\hbar_0/\hbar}, \qquad \hbar = \text{real} \tag{21}$$

related to a scaling of the diffusion constant $\hbar_0 = 2mD_0 \to \hbar = 2mD$. Upon performing such scaling, the ordinary *linear* Schrödinger equation in the variable ψ will appear to be *nonlinear* in the new scaled wavefunction ψ'

$$i\hbar\frac{\partial\psi'}{\partial t} = -\frac{\hbar^2}{2m}\frac{\hbar_0}{\hbar}\nabla^2\psi' + U\psi' - -\frac{\hbar^2}{2m}\left(1-\frac{\hbar_0}{\hbar}\right)\left(\vec{\nabla}\ln\psi'\right)^2\psi', \tag{22}$$

but this apparent nonlinearity is only an *artifact* of the change of variables (the scaling of ψ).

Notice that the latter (apparent) nonlinear equation, despite having the same form as the NLSE, obtained from a complex-diffusion constant, differs *crucially* in the actual values of the coefficients multiplying each of the terms. The NLSE has the complex coefficients $\alpha/\hbar$ (in the kinetic terms), and $-i\beta/\hbar$ (in the nonlinear logarithmic terms) with $\hbar = \alpha + i\beta = $ complex. However, the nonlinear equation obtained from a naive scaling involves *real* and *different* numerical coefficients than those present in the NLSE. Therefore, the genuine NLSE *cannot* be obtained by a naive scaling (redefinition) of the ψ and the diffusion constant.

Notice also that even if one scaled ψ by a complex exponent $\psi \to \psi^\lambda$ with $\lambda = \hbar_0/\hbar$ and $\hbar = $ complex, the actual numerical values in the apparent nonlinear equation, in general, would have still been different than those present in the NLSE. However, there is an actual equivalence, if, and only if, the scaling exponent $\lambda = \hbar_0/\hbar$ obeyed the condition:

$$\alpha = \hbar_0 \;\Rightarrow 1 - \frac{\hbar_0}{\hbar} = 1 - \frac{\alpha}{\hbar} = 1 - \frac{\hbar - i\beta}{\hbar} = i\,\frac{\beta}{\hbar} \tag{23}$$

in this very special case, the NLSE would be obtained from a linear Schrödinger equation after scaling the wavefunction $\psi \to \psi^\lambda$ with a complex exponent $\lambda = \hbar_0/\hbar = \alpha/\hbar$. In this very special and restricted case, the NLSE could be *linearized* by a scaling of the wavefunction with complex exponent.

From this analysis one infers, immediately, that if one defines the norm of the complex $\hbar$: $\|\hbar\| = \sqrt{\alpha^2+\beta^2} = \hbar_0$ to coincide precisely with the observed value $\hbar_0$ of Planck's constant, then $\alpha \neq \hbar_0$, $i\beta \neq \hbar - \hbar_0$ and, consequently, the

NLSE *cannot* be obtained from the ordinary (linear) Schrödinger equations after a naive scaling, with a complex exponent, $\psi \to \psi^\lambda = \psi^{\hbar_0/\hbar}$. Therefore, a complex diffusion constant $2mD = \hbar = \alpha + i\beta$, with the condition $2m\|D\| = \|\hbar\| = \sqrt{\alpha^2 + \beta^2} = \hbar_0$ (observed value of Planck's constant) ensures that the NLSE is not a mere artifact of the scaling of the wavefunction $\psi \to \psi^\lambda = \psi^{\hbar_0/\hbar}$ in the ordinary linear Schrödinger equation.

It is important to emphasize that the diffusion constant is always chosen to be related to Planck constant as follows: $2m\|D\| = \|\hbar\| = \hbar_0$ which is just the transition length from a fractal to a scale-independence non-fractal regime discussed by Nottale in numerous occasions. In the relativistic scale it is the Compton wavelength of the particle (say an electron): $\lambda_c = \hbar_0/(mc)$. In the nonrelativistic case it is the de Broglie wavelength of the electron.

Therefore, the NLSE based on a fractal Brownian motion with a complex valued diffusion constant $2mD = \hbar = \alpha + i\beta$ represents truly a new physical phenomenon and a hallmark of nonlinearity in QM. For other generalizations of QM see experimental tests of quaternionic QM (in the book by Adler [16]). Equation (18) is the fundamental NLSE of this work.

• A Fractal Scale Calculus description of our NLSE was developed later on by Cresson [20] who obtained, on a rigorous mathematical footing, the same functional form of our NLSE equation above (although with different complex numerical coefficients) by using Nottale's fractal scale-calculus that obeyed a quantum bialgebra. A review of our NLSE was also given later on by [25]. Our nonlinear wave equation originated from a complex-valued diffusion constant that is related to a complex-valued extension of Planck's constant. Hence, a fractal spacetime is deeply ingrained with *nonlinear* wave equations as we have shown and it was later corroborated by Cresson [20].

• Complex-valued viscosity solutions to the Navier-Stokes equations were also analyzed by Nottale leading to the Fokker-Planck equation. Clifford-valued extensions of QM were studied in [21] C-spaces (Clifford-spaces whose enlarged coordinates are polyvectors, i. e. antisymmetric tensors) that involved a Clifford-valued number extension of Planck's constant; i. e. the Planck constant was a hyper-complex number. Modified dispersion relations were derived from the underlying QM in Clifford-spaces that lead to faster than light propagation in ordinary spacetime but without violating causality in the more fundamental Clifford spaces. Therefore, one should not exclude the possibility of having complex-extensions of the Planck constant leading to non-linear wave equations associated with the Brownian motion of a particle in fractal spacetimes.

• Notice that the NLSE (34) obeys the homogeneity condition $\psi \to \lambda\psi$ for any constant λ. All the terms in the NLSE are scaled respectively by a factor λ. Moreover, our two parameters α, β are intrinsically connected to a complex Planck constant $\hbar = \alpha + i\beta$ such that $\|\hbar\| = \sqrt{\alpha^2 + \beta^2} = \hbar_0$ (observed Planck's constant) rather that being *ah-hoc* constants to be determined experimentally. Thus, the nonlinear QM equation derived from the fractal Brownian motion with complex-valued diffusion coefficient is intrinsically tied up with a non-Hermitian Hamiltonian and with complex-valued energy spectra [10].

• Despite having a non-Hermitian Hamiltonian we still could have eigenfunctions with real valued energies and momenta. Non-Hermitian Hamiltonians (pseudo-Hermitian) have captured a lot of interest lately in the so-called PT symmetric *complex* extensions of QM and QFT [27]. Therefore these ideas cannot be ruled out and they are the subject of active investigation nowadays.

3 Complex momenta, Weyl geometry, Bohm's potential and Fisher information

Despite that the interplay between Fisher Information and Bohm's potential has been studied by several authors [24] the importance of introducing a *complex* momentum $P_k = p_k + iA_k$ in order to fully understand the physical implications of Weyl's geometry in QM has been overlooked by several authors [24], [25]. We shall begin by reviewing the relationship between the Bohm's Quantum Potential and the Weyl curvature scalar of the Statistical ensemble of particle-paths (a fluid) associated to a single particle and that was developed by [22]. A Weyl geometric formulation of the Dirac equation and the nonlinear Klein-Gordon wave equation was provided by one of us [23]. Afterwards we will describe the interplay between Fisher Information and the Bohm's potential by introducing an action based on a *complex* momentum $P_k = p_k + iA_k$.

In the description of [22] one deals with a geometric derivation of the nonrelativistic Schrödinger Equation by relating the Bohm's quantum potential Q to the Ricci-Weyl scalar curvature of an ensemble of particle-paths associated to *one* particle. A quantum mechanical description of many particles is far more complex. This ensemble of particle paths resemble an Abelian *fluid* that permeates spacetime and whose ensemble density ρ affects the Weyl curvature of spacetime, which in turn, determines the geodesics of spacetime in guiding the particle trajectories. See [22], [23] for details.

Again a relation between the *relativistic* version of Bohm's potential Q and the Weyl-Ricci curvature exists but without the ordinary nonrelativistic probabilistic connections. In relativistic QM one does not speak of probability density to find a particle in a given spacetime point but instead one refers to the particle number current $J^\mu = \rho dx^\mu/d\tau$. In [22], [23] one begins with an ordinary Lagrangian associated with a point particle and whose statistical ensemble average over all particle-paths is performed only over the random initial data (configurations). Once the initial data is specified the trajectories (or rays) are completely determined by the

Hamilton-Jacobi equations. The statistical average over the random initial Cauchy data is performed by means of the ensemble density ρ. It is then shown that the Schrödinger equation can be derived after using the Hamilton-Jacobi equation in conjunction with the continuity equation and where the "quantum force" arising from Bohm's quantum potential Q can be related to (or described by) the Weyl geometric properties of space. To achieve this one defines the Lagrangian

$$L(q,\dot{q},t) = L_C(q,\dot{q},t) + \gamma\,(\hbar^2/m)\,R(q,t)\,, \tag{24}$$

where $\gamma = (1/6)(d-2)/(d-1)$ is a dimension-dependent numerical coefficient and R is the Weyl scalar curvature of the corresponding d-dimensional Weyl spacetime M where the particle lives.

Covariant derivatives are defined for contravariant vectors V^k: $V^k_{,ß} = \partial_i V^k - \Gamma^k_{im} V^m$ where the Weyl connection coefficients are composed of the ordinary Christoffel connection plus terms involving the Weyl gauge field of dilatations A_i. The curvature tensor R^i_{mkn} obeys the same symmetry relations as the curvature tensor of Riemann geometry as well as the Bianchi identity. The Ricci symmetric tensor R_{ik} and the scalar curvature R are defined by the same formulas also, viz. $R_{ik} = R^n_{ink}$ and $R = g^{ik} R_{ik}$

$$\begin{aligned} R_{\text{Weyl}} &= R_{\text{Riemann}} + \\ &+ (d-1)\left[(d-2)\,A_i\,A^i - \frac{2}{\sqrt{g}}\,\partial_i(\sqrt{g}A^i)\right], \end{aligned} \tag{25}$$

where R_{Riemann} is the ordinary Riemannian curvature defined in terms of the Christoffel symbols without the Weyl-gauge field contribution.

In the special case that the space is flat from the Riemannian point of view, after some algebra one can show that the Weyl scalar curvature contains only the Weyl gauge field of dilatations

$$R_{\text{Weyl}} = (d-1)(d-2)(A_k A^k) - 2\,(d-1)(\partial_k A^k)\,. \tag{26}$$

Now the Weyl geometrical properties are to be derived from physical principles so the A_i cannot be arbitrary but must be related to the distribution of matter encoded by the ensemble density of particle-paths ρ and can be obtained by the same (averaged) least action principle giving the motion of the particle. The minimum is to be evaluated now with respect to the class of all Weyl geometries having arbitrarily Weyl-gauge fields but with fixed metric tensor.

A variational procedure [22] yields a minimum for

$$A_i(q,t) = -\frac{1}{d-2}\,\partial_k(\log\rho) \Rightarrow F_{ij} = \partial_i A_j - \partial_j A_i = 0\,, \tag{27}$$

which means that the ensemble density ρ is Weyl-covariantly constant

$$\begin{aligned} \mathcal{D}_i\,\rho = 0 = \partial_i\,\rho + \omega(\rho)\,\rho\,A_i = 0 \;\Rightarrow \\ \Rightarrow\; A_i\,(q,t) = -\frac{1}{d-2}\,\partial_i(\log\rho)\,, \end{aligned} \tag{28}$$

where $\omega\,(\rho)$ is the Weyl weight of the density ρ. Since A_i is a *total* derivative the length of a vector transported from A to B along *different* paths changes by the *same amount*. Therefore, a vector after being transported along a *closed* path does *not* change its overall length. This is of fundamental importance to be able to solve in a satisfactory manner Einstein's objections to Weyl's geometry. If the lengths were to change in a path-dependent manner as one transports vectors from point A to point B, two atomic clocks which followed different paths from A to B will tick at *different* rates upon arrival at point B.

The continuity equation is

$$\frac{\partial\rho}{\partial t} + \frac{1}{\sqrt{g}}\,\partial_i\,(\sqrt{g}\,\rho\,v^i) = 0\,. \tag{29}$$

In this spirit one goes next to a geometrical derivation of the Schrödinger equation. By inserting

$$A_k = -\frac{1}{d-2}\,\frac{\partial\log\rho}{\partial x^k} \tag{30}$$

into

$$R_{\text{Weyl}} = (d-1)(d-2)(A_k A^k) - 2\,(d-1)\,\partial_k\,A^k \tag{31}$$

one gets for the Weyl scalar curvature, in the special case that the space is flat from the Riemannian point of view, the following expression

$$R_{\text{Weyl}} = \frac{1}{2\gamma\sqrt{\rho}}(\partial_i\,\partial^i\sqrt{\rho})\,, \tag{32}$$

which is precisely equal to the Bohm's Quantum potential up to numerical factors.

The Hamilton-Jacobi equation can be written as

$$\frac{\partial S}{\partial t} + H_C(q,S,t) - \gamma\left(\frac{\hbar^2}{2m}\right)R = 0\,, \tag{33}$$

where the effective Hamiltonian is

$$\begin{aligned} H_C - \gamma\left(\frac{\hbar^2}{m}\right)R &= \frac{1}{2m}\,g^{jk}p_j p_k + V - \gamma\,\frac{\hbar^2}{m}R = \\ &= \frac{1}{2m}\,g^{jk}\,\frac{\partial S}{\partial x^j}\,\frac{\partial S}{\partial x^k} + V - \gamma\,\frac{\hbar^2}{m}R\,. \end{aligned} \tag{34}$$

When the above expression for the Weyl scalar curvature (Bohm's quantum potential given in terms of the ensemble density) is inserted into the Hamilton-Jacobi equation, in conjunction with the continuity equation, for a momentum given by $p_k = \partial_k S$, one has then a set of two nonlinear coupled partial differential equations. After some straightforward algebra, one can verify that these two coupled differential equations equations will lead to the Schrödinger equation after the substitution $\Psi = \sqrt{\rho}\,e^{iS/\hbar}$ is made.

For example, when $d = 3$, $\gamma = 1/12$ and consequently, Bohm's quantum potential $Q = -(\hbar^2/12m)R$ (when R_{Riemann} is zero) becomes

$$R = \frac{1}{2\gamma\sqrt{\rho}}\partial_i\,g^{ik}\partial_k\sqrt{\rho} \sim \frac{1}{2\gamma}\,\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \Rightarrow Q = -\frac{\hbar^2}{2m}\,\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \tag{35}$$

as is should be and from the two coupled differential equations, the Hamilton-Jacobi and the continuity equation, they both reduce to the standard Schrödinger equation in flat space

$$i\hbar \frac{\partial \Psi(\vec{x},t)}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi(\vec{x}.t) + V\Psi(\vec{x},t) \tag{36}$$

after, and only after, one defines $\Psi = \sqrt{\rho}\, e^{iS/\hbar}$.

If one had a *curved* spacetime with a nontrivial metric one would obtain the Schrödinger equation in a curved spacetime manifold by replacing the Laplace operator by the Laplace-Beltrami operator. This requires, of course, to write the continuity and Hamilton Jacobi equations in a explicit covariant manner by using the covariant form of the divergence and Laplace operator [22], [23]. In this way, the geometric properties of space are indeed affected by the presence of the particle and in turn the alteration of geometry acts on the particle through the quantum force $f_i = \gamma(\hbar^2/m)\,\partial_i R$ which depends on the Weyl gauge potential A_i and its derivatives. It is this peculiar feedback between the Weyl geometry of space and the motion of the particle which recapture the effects of Bohm's quantum potential.

The formulation above from [22] was also developed for a derivation of the Klein-Gordon (KG) equation. The Dirac equation and Nonlinear Relativistic QM equations were found by [23] via an average action principle. The *relativistic* version of the Bohm potential (for signature $-,+,+,+$) can be written

$$Q \sim \frac{1}{m^2}\frac{(\partial_\mu\partial^\mu\sqrt{\rho})}{\sqrt{\rho}} \tag{37}$$

in terms of the D'Alambertian operator.

To finalize this section we will explain why the Bohm-potential/Weyl scalar curvature relationship in a flat spacetime

$$Q = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\,g^{ik}\partial_i\partial_k\sqrt{\rho} = \frac{\hbar^2 g^{ik}}{8m}\left(\frac{2\partial_i\partial_k\rho}{\rho} - \frac{\partial_i\rho\,\partial_k\rho}{\rho^2}\right) \tag{38}$$

encodes already the explicit connection between Fisher Information and the Weyl-Ricci scalar curvature R_{Weyl} (for Riemann flat spaces) after one realizes the importance of the *complex* momentum $P_k = p_k + iA_k$. This is typical of Electromagnetism after a minimal coupling of a charged particle (of charge e) to the $U(1)$ gauge field $\mathcal{A}_k$ is introduced as follows $\Pi_k = p_k + ie\mathcal{A}_k$. Weyl's initial goal was to unify Electromagnetism with Gravity. It was later realized that the gauge field of Weyl's dilatations A was *not* the same as the $U(1)$ gauge field of Electromagnetism $\mathcal{A}$.

Since we have reviewed the relationship between the Weyl scalar curvature and Bohm's Quantum potential, it is not surprising to find automatically a connection between Fisher information and Weyl Geometry after a *complex* momentum $P_k = p_k + iA_k$ is introduced. A complex momentum has already been discussed in previous sections within the context of fractal trajectories moving forwards and backwards in time by Nottale and Ord.

If ρ is defined over an d-dimensional manifold with metric g^{ik} one obtains a natural definition of the Fisher information associated with the ensemble density ρ

$$I = g^{ik}I_{ik} = \frac{g^{ik}}{2}\int \frac{1}{\rho}\frac{\partial\rho}{\partial y^i}\frac{\partial\rho}{\partial y^k}\,d^n y. \tag{39}$$

In the Hamilton-Jacobi formulation of classical mechanics the equation of motion takes the form

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\,g^{jk}\frac{\partial S}{\partial x^j}\frac{\partial S}{\partial x^k} + V = 0\,. \tag{40}$$

The momentum field p^j is given by $p^j = g^{jk}(\partial S/\partial x^k)$. The ensemble probability density of particle-paths $\rho(t, x^\mu)$ obeys the normalization condition $\int d^n x\,\rho = 1$. The continuity equation is

$$\frac{\partial\rho}{\partial t} + \frac{1}{m}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}\,\rho\, g^{jk}\frac{\partial S}{\partial x^k}\right) = 0\,. \tag{41}$$

These equations completely describe the motion and can be derived from the action

$$S = \int \rho\left(\frac{\partial S}{\partial t} + \frac{1}{2m}\,g^{jk}\frac{\partial S}{\partial x^j}\frac{\partial S}{\partial x^k} + V\right) dt\, d^n x \tag{42}$$

using fixed endpoint variation in S and ρ.

The Quantization via the Weyl geometry procedure is obtained by defining the *complex* momentum in terms of the Weyl gauge field of dilatations A_k as $P_k = p_k + ieA_k$ and constructing the modified Hamiltonian in terms of the norm-squared of the *complex* momentum $P^k P^*_k$ as follows

$$H_{\text{Weyl}} = \frac{g^{jk}}{2m}\left[(p_j + ieA_j)(p_k - ieA_k)\right] + V. \tag{43}$$

The *modified* action is now:

$$S_{\text{Weyl}} = \int dt\, d^n x\left[\frac{\partial S}{\partial t} + \frac{g^{jk}}{2m}(p_j + ieA_j)(p_k - ieA_k) + V\right]. \tag{44}$$

The relationship between the Weyl gauge potential and the ensemble density ρ was

$$A_k \sim \frac{\partial \log(\rho)}{\partial x^k} \tag{45}$$

the proportionality factors can be re-absorbed into the coupling constant e as follows $P_k = p_k + ieA_k = p_k + i\,\partial_k(\log\rho)$. Hence, when the spacetime metric is flat (diagonal) $g^{jk} = \delta^{jk}$, S_{Weyl} becomes

$$S_{\text{Weyl}} = \int dt\, d^n x\,\frac{\partial S}{\partial t} + \frac{g^{jk}}{2m}\left[\left(\frac{\partial S}{\partial x^j} + i\frac{\partial\log(\rho)}{\partial x^j}\right)\times \times\left(\frac{\partial S}{\partial x^k} - i\frac{\partial\log(\rho)}{\partial x^k}\right)\right] + V = \int dt\, d^n x\left[\frac{\partial S}{\partial t} + V + + \frac{g^{jk}}{2m}\left(\frac{\partial S}{\partial x^j}\right)\left(\frac{\partial S}{\partial x^k}\right)\right] + \frac{1}{2m}\int dt\, d^n x\left[\frac{1}{\rho}\frac{\partial\rho}{\partial x^k}\right]^2. \tag{46}$$

The expectation value of S_{Weyl} is

$$< S_{\text{Weyl}} > \; = \; < S_C > + S_{\text{Fisher}} = \int dt\, d^n x\, \rho \left[\frac{\partial S}{\partial t} + \right.$$
$$\left. + \frac{g^{jk}}{2m} \left(\frac{\partial S}{\partial x^j} \right) \left(\frac{\partial S}{\partial x^k} \right) + V \right] + \frac{1}{2m} \int dt\, d^n x\, \rho \left[\frac{1}{\rho} \frac{\partial \rho}{\partial x^k} \right]^2 . \tag{47}$$

This is how we have reproduced the Fisher Information expression directly from the last term of $< S_{\text{Weyl}} >$:

$$S_{\text{Fisher}} \equiv \frac{1}{2m} \int dt\, d^n x\, \rho \left[\frac{1}{\rho} \frac{\partial \rho}{\partial x^k} \right]^2 . \tag{48}$$

An Euler variation of the expectation value of the action $< S_{\text{Weyl}} >$ with respect to the ρ yields:

$$\frac{\partial S}{\partial t} + \frac{\delta < S_{\text{Weyl}} >}{\delta \rho} - \partial_j \left(\frac{\delta < S_{\text{Weyl}} >}{\delta (\partial_j \rho)} \right) = 0 \;\Rightarrow \tag{49}$$

$$\frac{\partial S}{\partial t} + V + \frac{1}{2m} g^{jk} \left[\frac{\partial S}{\partial x^j} \frac{\partial S}{\partial x^k} + \right.$$
$$\left. + \left(\frac{1}{\rho^2} \frac{\partial \rho}{\partial x^j} \frac{\partial \rho}{\partial x^k} - \frac{2}{\rho} \frac{\partial^2 \rho}{\partial x^j \partial x^k} \right) \right] = 0 . \tag{50}$$

Notice that the last term of the Euler variation

$$\frac{1}{2m} g^{jk} \left[\left(\frac{1}{\rho^2} \frac{\partial \rho}{\partial x^j} \frac{\partial \rho}{\partial x^k} - \frac{2}{\rho} \frac{\partial^2 \rho}{\partial x^j \partial x^k} \right) \right] \tag{51}$$

is precisely the same as the Bohm's quantum potential , which in turn, is proportional to the Weyl scalar curvature. If the continuity equation is implemented at this point one can verify once again that the last equation is equivalent to the Schrödinger equation after the replacement $\Psi = \sqrt{\rho}\, e^{iS/\hbar}$ is made.

Notice that in the Euler variation variation of $< S_{\text{Weyl}} >$ w. r. t the ρ one must include those terms involving the derivatives of ρ as follows

$$-\partial_j \left(\frac{\delta \left[\rho (\partial_k \rho / \rho)^2 \right]}{\delta (\partial_j \rho)} \right) = -\frac{1}{\rho} \partial_j \left(\frac{\delta (\partial_k \rho)^2}{\delta (\partial_j \rho)} \right) = -\frac{2}{\rho} \partial_j \partial^j \rho . \tag{52}$$

This explains the origins of all the terms in the Euler variation that yield Bohm's quantum potential.

Hence, to conclude, we have shown how the *last* term of the Euler variation of the averaged action $< S_{\text{Weyl}} >$, that automatically incorporates the Fisher Information expression after a *complex* momentum $P_k = p_k + i\partial_k (\log \rho)$ is introduced via the Weyl gauge field of dilations $A_k \sim -\partial_k \log \rho$, generates once again Bohm's potential:

$$Q \sim \left(\frac{1}{\rho^2} \frac{\partial \rho}{\partial x^j} \frac{\partial \rho}{\partial x^k} - \frac{2}{\rho} \frac{\partial^2 \rho}{\partial x^j \partial x^k} \right) . \tag{53}$$

To conclude, the Quantization of a particle whose Statistical ensemble of particle-paths permeate a spacetime background endowed with a Weyl geometry allows to construct a *complex* momentum $P_k = \partial_k S + i\partial_k (\log \rho)$ that yields automatically the Fisher Information S_{Fisher} term. The latter Fisher Information term is crucial in generating Bohm's quantum potential Q after an Euler variation of the expectation value of the $<S_{\text{Weyl}}>$ with respect to the ρ is performed. Once the Bohm's quantum potential is obtained one recovers the Schrödinger equation after implementing the continuity equation and performing the replacement $\Psi = \sqrt{\rho}\, e^{iS/\hbar}$. This completes the relationship among Bohm's potential, the Weyl scalar curvature *and* Fisher Information *after* introducing a *complex* momentum.

4 Concluding remarks

Based on Nottale and Ord's formulation of QM from first principles; i. e. from the fractal Brownian motion of a massive particle we have derived explicitly a nonlinear Schrödinger equation. Despite the fact that the Hamiltonian is not Hermitian, real-valued energy solutions exist like the plane wave and soliton solutions found in the free particle case. The remarkable feature of the fractal approach versus *all* the Nonlinear QM equation considered so far is that the Quantum Mechanical energy functional coincides precisely with the field theory one.

It has been known for some time, see Puskarz [8], that the expression for the energy functional in nonlinear QM does *not* coincide with the QM energy functional, nor it is unique. The classic Gross-Pitaveskii NLSE (of the 1960's), based on a quartic interaction potential energy, relevant to Bose-Einstein condensation, contains the nonlinear cubic terms in the Schrödinger equation, after differentiation, $(\psi^* \psi)\psi$. This equation does not satisfy the Weinberg homogeneity condition [9] and also the energy functional differs from the E_{QM} by factors of two.

However, in the fractal-based NLSE there is no discrepancy between the quantum-mechanical energy functional and the field theory energy functional. Both are given by

$$H^{\text{NLSE}}_{\text{fractal}} = -\frac{\hbar^2}{2m} \frac{\alpha}{\hbar} \psi^* \nabla^2 \psi + U \psi^* \psi -$$
$$- i \frac{\hbar^2}{2m} \frac{\beta}{\hbar} \psi^* (\vec{\nabla} \ln \psi)^2 \psi . \tag{54}$$

This is why we push forward the NLSE derived from the fractal Brownian motion with a complex-valued diffusion coefficient. Such equation does admit plane-wave solutions with the dispersion relation $E = \vec{p}^{\,2}/(2m)$. It is not hard to see that after inserting the plane wave solution into the fractal-based NLSE we get (after setting $U = 0$),

$$E = \frac{\hbar^2}{2m} \frac{\alpha}{\hbar} \frac{\vec{p}^{\,2}}{\hbar^2} + i \frac{\beta}{\hbar} \frac{\vec{p}^{\,2}}{2m} = \frac{\vec{p}^{\,2}}{2m} \frac{\alpha + i\beta}{\hbar} = \frac{\vec{p}^{\,2}}{2m} , \tag{55}$$

since $\hbar = \alpha + i\beta$. Hence, the plane-wave *is* a solution to our fractal-based NLSE (when $U = 0$) with a real-valued energy and has the correct energy-momentum dispersion relation.

Soliton solutions, with real-valued energy (momentum) are of the form

$$\psi \sim \big[F(x-vt) + iG(x-vt)\big] e^{ipx/\hbar - iEt/\hbar}\,, \qquad (56)$$

with F, G two functions of the argument $x - vt$ obeying a coupled set of two nonlinear differential equations.

It is warranted to study solutions when one turns-on an external potential $U \neq 0$ and to generalize this construction to the Quaternionic Schrödinger equation [16] based on the Hydrodynamical Nonabelian-fluid Madelung's formulation of QM proposed by [26]. And, in particular, to explore further the consequences of the Non-Hermitian Hamiltonian (pseudo-Hermitian) associated with our NLSE (34) within the context of the so-called PT symmetric *complex* extensions of QM and QFT [27]. Arguments why a quantum theory of gravity should be nonlinear have been presented by [28] where a *different* non-linear Schrödinger equation, but with a similar logarithmic dependence, was found. This equation [28] is also similar to the one proposed by Doebner and Goldin [29] from considerations of unitary representations of the diffeomorphism group.

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