Correct Linearization of Einstein's Equations

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Routinely, Einstein's equations are be reduced to a wave form (linearly independent of the second derivatives of the space metric) in the absence of gravitation, the space rotation and Christoffel's symbols. As shown herein, the origin of the problem is the use of the general covariant theory of measurement. Herein the wave form of Einstein's equations is obtained in terms of Zelmanov's chronometric invariants (physically observable projections on the observer's time line and spatial section). The equations so obtained depend solely upon the second derivatives, even for gravitation, the space rotation and Christoffel's symbols. The correct linearization proves that the Einstein equations are completely compatible with weak waves of the metric.

1 Introduction

Gravitational waves are routinely considered as weak waves of the space metric, whereby, one takes a Galilean metric $g^{(0)}_{\alpha\beta}$, whose components are $g^{(0)}_{00}=1$, $g^{(0)}_{0i}=0$, $g^{(0)}_{ik}=-\delta_{ik}$, and says: because gravitating matter is connected to the field of the metric tensor $g_{\alpha\beta}$ by Einstein's equations*

$$R_{\alpha\beta}-\frac{1}{2}\,g_{\alpha\beta}R=-\kappa\,T_{\alpha\beta}+\lambda g_{\alpha\beta}\,,\quad \kappa=\text{const}>0\,,$$

gravitational waves are weak perturbations $\zeta_{\alpha\beta}$ of the Galilean metric. Thus the common metric, consisting of the initially undeformed and wave parts, is $g_{\alpha\beta}=g^{(0)}_{\alpha\beta}+\zeta_{\alpha\beta}$.

According to the theory of partial differential equations, a wave of a field is a Hadamard break [1] in the derivatives of the field function along the hypersurface of the field equations (the wave front). The first derivative of a function at a point determines a direction tangential to it, while the second derivative determines a normal direction. Thus, if a surface in a tensor field is the front of the field wave, the second derivatives of this tensor have breaks there. It is possible to prove in relation to this case in a Riemannian space with the metric $g_{\alpha\beta}$, that d'Alembert's operator $\Box=g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta$ of this field equals zero†. For instance, the wave field of a tensor $Q_{\mu\nu}$ is characterized by the d'Alembert equations $\Box\,Q_{\mu\nu}=0$.

We can apply the d'Alembert operator to any tensor field and equate it to be zero. For this reason any claims that waves of the space metric cannot exist are wrong, even from the purely mathematical viewpoint, independently of those deductions that the authors of those claims adduced.

So, the front of weak wave perturbations $\zeta_{\alpha\beta}$ of a Galilean metric $g^{(0)}_{\alpha\beta}$ is determined by breaks in their second derivatives, while the wave field $\zeta_{\alpha\beta}$ itself is characterized by the d'Alembert equations

$$\Box\,\zeta_{\alpha\beta}=0\,.$$

If the left side of the Einstein equations for the common metric $g_{\alpha\beta}=g^{(0)}_{\alpha\beta}+\zeta_{\alpha\beta}$ reduced to $\Box\,\zeta_{\alpha\beta}$,‡ the equations could be reduced to the form

$$a\,\Box\,\zeta_{\alpha\beta}=-\kappa\,T_{\alpha\beta}+\lambda g_{\alpha\beta}\,,\qquad\text{where}\quad a=\text{const}\,,$$

which, in the absence of matter, become the wave equations $\Box\,\zeta_{\alpha\beta}=0$, meaning that the perturbations $\zeta_{\alpha\beta}$ are waves.

As one calculates the left side of the Einstein equations for the common metric, he obtains a large number of terms where only one is $\Box\,\zeta_{\alpha\beta}$ with a numerical coefficient. Thus one concludes: the Einstein equations are non-linear with respect to the second derivatives of $\zeta_{\alpha\beta}$.

In order to prove gravitational waves, theory should lead to cancellation of all the non-linear terms, as argued by Eddington [2], and Landau and Lifshitz [3]. This process is so-called the *linearization* of the Einstein equations.

2 Problems with the linearization

There is much literature about why the non-linear terms can be cancelled (see Lichnerowicz [4] or Zakharov [5] for details). All the reasons depend upon one initial factor: the theory of measurements we use.

We know two theories of measurements in General Relativity: Einstein's theory of measurements and Zelmanov's theory of physically observable quantities. The first one was built by Einstein in the 1910's. Following him§, we consider the space-time volume of nearby events in order to find a particular reference frame satisfying the properties of our real laboratory. We then express our general covariant equations in terms of the chosen reference frame. Some terms drop out, because of the properties of the chosen reference frame. Briefly, as one calculates the Ricci tensor $R_{\alpha\beta}=g^{\mu\nu}R_{\alpha\mu\nu\beta}$ by the contraction of the Riemann-Christoffel tensor

*We write the Einstein equations in the main form containing the λ-term, because our consideration is outside a discussion of the λ-term.

†Note that the d'Alembert operator consists of the second derivatives.

‡Actually, this problem is to reduce the Ricci tensor for the common metric $g_{\alpha\beta}=g^{(0)}_{\alpha\beta}+\zeta_{\alpha\beta}$ to $\Box\,\zeta_{\alpha\beta}$.

§Einstein gave his theory of measurements partially in many papers. You can see the complete theory in Synge's book [6], for instance.

$$R_{\alpha\mu\nu\beta} = -\Gamma^{\sigma}_{\mu\beta}\,\Gamma_{\alpha\nu,\sigma} + \Gamma^{\sigma}_{\mu\nu}\,\Gamma_{\alpha\beta,\sigma} + \\ + \frac{1}{2}\left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta} + \frac{\partial^2 g_{\alpha\beta}}{\partial x^\nu \partial x^\mu} - \frac{\partial^2 g_{\alpha\nu}}{\partial x^\mu \partial x^\beta} - \frac{\partial^2 g_{\mu\beta}}{\partial x^\alpha \partial x^\nu}\right)$$

for $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \zeta_{\alpha\beta}$ (see §105 in [3]), he can reduce it to

$$R_{\alpha\beta} = \frac{1}{2}\, g^{(0)\mu\nu}\,\frac{\partial^2 \zeta_{\alpha\beta}}{\partial x^\mu \partial x^\nu} = \frac{1}{2}\,\Box\,\zeta_{\alpha\beta}$$

and the left side of the Einstein equations to $\Box\,\zeta_{\alpha\beta}$, only if:

1. The reference frame is free of forces of gravity;
2. The reference frame is free of rotation;
3. Christoffel's symbols $\Gamma^{\alpha}_{\mu\nu}$, containing the inhomogeneity of space, are all zero.

Of course, we can find a reference frame where the gravitational potential, the space rotation, and the Christoffel symbols are zero at a given point*. However they cannot be reduced to zero in an area. Moreover, a gravitational wave detector consists of two bodies located far away from each other. In a Weber solid-body detector the distance is several metres, while in a laser interferometer the distance can take even millions of kilometres, as LISA in a solar orbit. It is wrong to interpret any of those as points. So, gravitational forces, the space rotation or the Christoffel symbols cannot be obviated in the equations. This is the main reason why:

> By the methods of Einstein's theory of measurements, the Einstein equations cannot be mathematically correctly linearized with respect to the second derivatives of the weak perturbations $\zeta_{\alpha\beta}$ of the space metric.

Some understand this incompatibility to mean that General Relativity does not permit weak waves of the metric.

This is absolutely wrong, even from the purely mathematical viewpoint: the d'Alembert operator $\Box = g^{\alpha\beta}\,\nabla_\alpha \nabla_\beta$ may be applied to any tensor field, the field of the weak perturbations $\zeta_{\alpha\beta}$ of the metric included, and equated to zero.

This obvious incompatibility can arise for one or both of the following reasons:

1. Einstein's equations in their current form are insufficient to describe our real world;
2. Einstein's theory of measurements is inadequate for the four-dimensional pseudo-Riemannian space.

Einstein's equations were born of his intuition, only the left side thereof is derived from the geometry. However main experimental tests of General Relativity, proceeding from the equations, verify the theory. So, the equations are adequate for describing our real world to within a first approximation.

At the same time, Einstein's theory of measurements has many deficiencies. There are no clear methods for recognition of physically observable components of a tensor field. It set up so that the three-dimensional components of a world-vector field compose its spatially observable part, while the time component is its scalar potential. However this problem becomes confused for a tensor of higher rank, because it has time, spatial, and mixed (space-time) components. There are also other drawbacks (see [8], for instance).

The required mathematical methods have been found by Zelmanov, who, in 1944, fused them into a complete theory of physically observable quantities [9, 10, 11].

*See §7 *Special Reference Frames* in Petrov's book [7].

3 The theory of physically observable quantities

According to Zelmanov, each observer has his own spatial section, set up by a coordinate net spanned over his real reference rest-body and extended far away with its gravitational field. The net is replete with a system of synchronized clocks†. Physically observed by him are projections of world-quantities onto his time line and spatial section, made by the projection operators $b^\alpha = \frac{dx^\alpha}{ds}$ and $h_{\alpha\beta} = -g_{\alpha\beta} + b_\alpha b_\beta$. Chr.inv.-projections of a world-vector Q^α are $b_\alpha Q^\alpha = \frac{Q_0}{\sqrt{g_{00}}}$ and $h^i_\alpha Q^\alpha = Q^i$, while those of a 2nd rank world-tensor $Q^{\alpha\beta}$ are $b^\alpha b^\beta Q_{\alpha\beta} = \frac{Q_{00}}{g_{00}}$, $h^{i\alpha} b^\beta Q_{\alpha\beta} = \frac{Q^i_0}{\sqrt{g_{00}}}$, $h^i_\alpha h^k_\beta Q^{\alpha\beta} = Q^{ik}$. Physically observable properties of the space are determined by the non-commutativity of the chr.inv.-operators $\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}$ and $\frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2}\,v_i\,\frac{{}^*\partial}{\partial t}$, and the fact that the chr.inv.-metric tensor $h_{ik} = -g_{ik} + \frac{1}{c^2}\,v_i v_k$ may not be stationary. They are the chr.inv.-quantities: the gravitational inertial force F_i, the space rotation tensor A_{ik}, and the space deformational rates D_{ik}

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \qquad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,,$$

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right), \quad v_i = -\frac{c\,g_{0i}}{\sqrt{g_{00}}}\,,$$

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, \quad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}\,, \quad D = D^k_k = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,,$$

where w is gravitational potential, v_i is the linear velocity of the space rotation, $h = \det\|h_{ik}\|$, and $\sqrt{-g} = \sqrt{h}\,\sqrt{g_{00}}$. The chr.inv.-Christoffel symbols $\Delta^i_{jk} = h^{im}\,\Delta_{jk,m}$ are built like the usual $\Gamma^\alpha_{\mu\nu} = g^{\alpha\sigma}\,\Gamma_{\mu\nu,\sigma}$, using h_{ik} instead of $g_{\alpha\beta}$.

By analogy with the Riemann-Christoffel curvature tensor, Zelmanov derived the chr.inv.-curvature tensor‡

$$C_{lkij} = \frac{1}{4}\left(H_{lkij} - H_{jkil} + H_{klji} - H_{iljk}\right),$$

from which the contraction $C_{kj} = C_{kij\cdot}^{\cdots i} = h^{im}\,C_{kimj}$ gives the chr.inv.-scalar observable curvature $C = C^j_j = h^{lj}\,C_{lj}$.

†Projections onto such a spatial section are independent of transformations of the time coordinate — they are *chronometric invariants*.

‡Here $H_{lki\cdot}^{\cdots j} = \frac{{}^*\partial \Delta^j_{il}}{\partial x^k} - \frac{{}^*\partial \Delta^j_{kl}}{\partial x^i} + \Delta^m_{il}\Delta^j_{km} - \Delta^m_{kl}\Delta^j_{im}$.

4 Correct linearization of Einstein's equations

We now show that Einstein's equations expressed with physically observable quantities may be linearized without problems; proof that waves of weak perturbations of the space metric are fully compatible with the Einstein equations.

Zelmanov already deduced [9] the Einstein equations in chr.inv.-components (the chr.inv.-Einstein equations) in the absence of matter: —

$$\frac{{}^*\partial D}{\partial t} + D_{jl}D^{jl} + A_{jl}A^{lj} + \left({}^*\nabla_j - \frac{1}{c^2}F_j\right)F^j = 0\,,$$

$${}^*\nabla_j\left(h^{ij}D - D^{ij} - A^{ij}\right) + \frac{2}{c^2}F_jA^{ij} = 0\,,$$

$$\frac{{}^*\partial D_{ik}}{\partial t} - \left(D_{ij}+A_{ij}\right)\left(D^j_k + A^{\cdot j}_{k\cdot}\right) + DD_{ik} + 3A_{ij}A^{\cdot j}_{k\cdot} + \\ + \frac{1}{2}\left({}^*\nabla_iF_k + {}^*\nabla_kF_i\right) - \frac{1}{c^2}F_iF_k - c^2C_{ik} = 0\,,$$

where Zelmanov's ${}^*\nabla_k$ denotes the chr.inv.-derivative*.

The components of the metric $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \zeta_{\alpha\beta}$, consisting of a Galilean metric and its weak perturbations, are†

$$\begin{aligned} g_{00} &= 1 + \zeta_{00}\,, & g_{0i} &= \zeta_{0i}\,, & g_{ik} &= -\delta_{ik} + \zeta_{ik}\,,\\ g^{00} &= 1 - \zeta^{00}, & g^{0i} &= -\zeta^{0i}, & g^{ik} &= -\delta^{ik} - \zeta^{ik},\\ h_{ik} &= \delta_{ik} - \zeta_{ik}, & h^i_k &= \delta^i_k\,, & h^{ik} &= \delta^{ik} + \zeta^{ik}. \end{aligned}$$

Because $\zeta_{\alpha\beta}$ are weak, the products of their components or derivatives vanish. In such a case,

$$F_i = \frac{c}{1+\zeta_{00}}\left(\frac{\partial\zeta_{0i}}{\partial t} - \frac{c}{2}\frac{\partial\zeta_{00}}{\partial x^i}\right),$$

$$A_{ik} = \frac{c}{\sqrt{1+\zeta_{00}}}\left(\frac{\partial\zeta_{0i}}{\partial x^k} - \frac{\partial\zeta_{0k}}{\partial x^i}\right),$$

$$D_{ik} = -\frac{1}{2\sqrt{1+\zeta_{00}}}\frac{\partial\zeta_{ik}}{\partial t}\,,\qquad D = h^{ik}D_{ik} = \delta^{ik}D_{ik}\,,$$

$$C_{imnk} = \frac{\partial^2\zeta_{mk}}{\partial x^i\partial x^n} + \frac{\partial^2\zeta_{in}}{\partial x^m\partial x^k} - \frac{\partial^2\zeta_{mn}}{\partial x^i\partial x^k} - \frac{\partial^2\zeta_{ik}}{\partial x^m\partial x^n}\,.$$

After some algebra, we obtain the chr.inv.-Einstein equations for the metric $g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \zeta_{\alpha\beta}$:

$$\frac{1}{c^2\left(1+\zeta_{00}\right)}\frac{\partial^2\zeta}{\partial t^2} + \frac{\delta^{km}}{\left(1+\zeta_{00}\right)}\left(\frac{\partial^2\zeta_{00}}{\partial x^k\partial x^m} - \frac{2}{c}\frac{\partial^2\zeta_{0m}}{\partial x^k\partial t}\right) = 0\,,$$

$$\frac{\delta^{ij}}{c^2\sqrt{1+\zeta_{00}}}\frac{\partial^2\zeta}{\partial x^j\partial t} - \frac{1}{c^2\sqrt{1+\zeta_{00}}}\frac{\partial^2\zeta^{ij}}{\partial x^j\partial t} + \\ + \frac{2\delta^{im}\delta^{jn}}{c\sqrt{1+\zeta_{00}}}\left(\frac{\partial^2\zeta_{0m}}{\partial x^j\partial x^n} - \frac{\partial^2\zeta_{0n}}{\partial x^j\partial x^m}\right) = 0\,,$$

$$\frac{1}{c^2\left(1+\zeta_{00}\right)}\frac{\partial^2\zeta_{ik}}{\partial t^2} - \frac{1}{c\left(1+\zeta_{00}\right)}\left(\frac{\partial^2\zeta_{0k}}{\partial x^i\partial t} - \frac{\partial^2\zeta_{0i}}{\partial x^k\partial t}\right) + \\ + 2\delta^{mn}\left(\frac{\partial^2\zeta_{mk}}{\partial x^i\partial x^n} + \frac{\partial^2\zeta_{in}}{\partial x^m\partial x^k} - \frac{\partial^2\zeta_{mn}}{\partial x^i\partial x^k} - \frac{\partial^2\zeta_{ik}}{\partial x^m\partial x^n}\right) = 0\,.$$

Note that the obtained equations are functions of only the second derivatives of the weak perturbations of the space metric. So, the Einstein equations have been linearized, even in the presence of gravitational inertial forces and the space rotation. This implies: —

> By the methods of Zelmanov's mathematical theory of chronometric invariants (physically observable quantities), **the Einstein equations are linearized in a mathematically correct way**, i. e. without the assumption of a specific reference frame where there are no gravitational forces or the space rotation.

This is the mathematical proof to the statement: —

> Waves of the weak perturbations of the space metric are fully compatible with the Einstein equations.

*So ${}^*\nabla_kQ^i = \frac{{}^*\partial Q^i}{\partial x^k} + \Delta^i_{mk}Q^m$ and ${}^*\nabla_kQ_i = \frac{{}^*\partial Q_i}{\partial x^k} - \Delta^m_{ik}Q_m$ are the chr.inv.-derivatives of a chr.inv.-vector Q^i.

†The contravariant tensor $g^{\alpha\beta}$, determined by the main property $g_{\alpha\sigma}g^{\sigma\beta} = \delta^\beta_\alpha$ of the fundamental metric tensor as $\left(g^{(0)}_{\alpha\sigma} + \zeta_{\alpha\sigma}\right)g^{\sigma\beta} = \delta^\beta_\alpha$, is $g^{\alpha\beta} = g^{(0)\alpha\beta} - \zeta^{\alpha\beta}$, while its determinant is $g = g^{(0)}(1+\zeta)$. This is easy to check, taking into account that, because the values of the weak corrections $\zeta_{\alpha\beta}$ are infinitesimal, their products vanish; while we may move indices in $\zeta_{\alpha\beta}$ by the Galilean metric tensor $g^{(0)}_{\alpha\beta}$.

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