Exact Theory of a Gravitational Wave Detector. New Experiments Proposed

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We deduce exact solutions to the deviation equation in the cases of both free and spring-connected particles. The solutions show that gravitational waves may displace particles in a two-particle system only if they are in motion with respect to each other or the local space (there is no effect if they are at rest). We therefore propose a new experimental statement for the detection of gravitational waves: use a suspended solid-body detector self-vibrating so that there are relative oscillations of its butt-ends. Or, in another way: use a free-mass detector fitted with suspended, vibrating mirrors. Such systems may have a relative displacement of the butt-ends and a time shift in the butt-ends, produced by a falling gravitational wave.

The authors dedicate this paper to the memory of Joseph Weber, who pioneered the detection of gravitational waves.

1 Introduction

As Borissova recently showed [1] by the Synge equation for deviating geodesic lines and the Synge-Weber equation for deviating non-geodesics, Weber's experimental statement on gravitational waves [2] is inadequate. His conclusions were not based upon an exact solution to the equations, but on an approximate analysis of what could be expected. Weber expected that a plane weak wave of the space metric (gravitational wave) may displace two particles at rest with respect to one another. The Weber equations and their solutions formulated in terms of the physically observable quantities show instead that gravitational waves cannot displace resting particles; some effect may be produced only if the particles are in motion.

Here we deduce exact solutions to both the Synge equation and the Synge-Weber equation (the exact theory to free-mass and solid-body detectors). The exact solutions show that we may alter the construction of both solid-body and free-mass detectors so that they may register oscillations produced by gravitational waves. Weber most probably detected them as claimed in 1968 [3, 4, 5], as his room-temperature solid-body pigs may have their own relative oscillations of the butt-ends, whereas the oscillations are inadvertently suppressed as noise in the detectors developed by his all followers, who have had no positive result in over 35-years.

2 Main equations of the theory

We consider two cases of a simple system consisting of two particles, either free or connected by a spring. A falling gravitational wave as a wave of the space metric deforming the space should produce some effect in such a system. Therefore we call such a system a gravitational wave detector.

We will determine the effect produced by a gravitational wave in both kinds of the two-particle systems.

If the particles are connected by a non-gravitational force Φ^α, they move along neighbouring non-geodesic world-lines, according to the non-geodesic equations of motion*

$$\frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} U^\mu U^\nu = \frac{\Phi^\alpha}{m_0 c^2}\,, \tag{1}$$

while relative oscillations of the world-lines (particles) are described by the so-called Synge-Weber equation† [2]

$$\frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha}_{\cdot\beta\gamma\delta}\, U^\beta U^\delta \eta^\gamma = \frac{1}{m_0 c^2}\frac{\mathrm{D}\Phi^\alpha}{dv}dv\,. \tag{2}$$

If two neighbouring particles are free ($\Phi^\alpha = 0$), they move along neighbouring geodesic lines, according to the geodesic equations of motion

$$\frac{dU^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} U^\mu U^\nu = 0\,, \tag{3}$$

while relative oscillations of the geodesics (particles) are given by the so-called Synge equations [6]

$$\frac{\mathrm{D}^2\eta^\alpha}{ds^2} + R^{\alpha}_{\cdot\beta\gamma\delta}\, U^\beta U^\delta \eta^\gamma = 0\,. \tag{4}$$

A solution to the deviation equations (4) or (2) gives the deviation $\eta^\alpha = (\eta^0, \eta^1, \eta^2, \eta^3)$ between the particles in the acting gravitational field. Because the field is unspecified in the equations (it is hidden in the formula for the metric ds), the equations allow the deviation to be described in both regular and wave fields of gravitation. Thus to determine

*Here $U^\alpha = \frac{dx^\alpha}{ds}$ is the four-dimensional velocity vector of the particle, tangential to its world-line. It is a unit world-vector: $U_\alpha U^\alpha = 1$. The space-time interval ds along the world-line is used as a parameter for differentiation, m_0 is the rest-mass of the particle, $\Gamma^\alpha_{\mu\nu}$ are Christoffel's symbols of the 2nd kind.

†Here $\frac{\mathrm{D}}{ds}$ is the absolute (covariant) differentiation operator; $R^{\alpha}_{\cdot\beta\gamma\delta}$ is the Riemann-Christoffel curvature tensor; $\eta^\alpha = \frac{\partial x^\alpha}{\partial v}dv$ is the relative deviation vector of the particles; v is a parameter having the same numerical value along a neighbouring world-line, while dv is the difference between its values in the world-lines.

how a gravitational wave causes two test-particles to deviate from one another, we should use the metric ds for this wave field and obtain exact solutions to the deviation equation.

Currently, two main kinds of gravitational wave detectors are presumed:

1. Weber's solid-body detector — a freely suspended bulky cylindrical pig, approximated by two masses connected by a spring (i. e. non-gravitational force). Oscillations of the butt-ends of the pig in the field of a falling gravitational wave are formulated by the Synge-Weber equation of deviating non-geodesics;
2. A free-mass detector, consisting of two freely suspended mirrors, distantly separated. Each mirror is fitted with a laser range-finder for producing measurements of the distance between them. Oscillations of the mirrors under a falling gravitational wave formulated by the Synge equation of deviating geodesics.

Both detectors have a common theory — the Synge-Weber equation, in comparison to the Synge equation, has just the non-zero right side with a force Φ^α connecting the particles. We may solve them using the same method. Before doing that however, we analyse Weber's approach to the main equations and his simplifications.

3 Weber's approach and criticism thereof

Weber proceeded from the proposition that a falling gravitational wave should deform a solid-body pig, represented by a system of two particles connected by a spring. He proposed the relative displacement of the particles η^α consisting of a "basic" displacement r^α (covariantly constant) and an infinitely small relative displacement ζ^α in the butt-ends of the cylinder caused by a falling gravitational wave

$$\eta^\alpha = r^\alpha + \zeta^\alpha, \qquad \zeta^\alpha \ll r^\alpha, \qquad \frac{\mathrm{D}r^\alpha}{ds} = 0\,. \tag{5}$$

Thus the non-geodesic deviation equation is

$$\frac{\mathrm{D}^2\zeta^\alpha}{ds^2} + R^{\alpha}_{\cdot\beta\gamma\delta}\,U^\beta U^\delta\left(r^\gamma + \zeta^\gamma\right) = \frac{\Phi^\alpha}{m_0c^2}\,, \tag{6}$$

which he transformed to*

$$\frac{\mathrm{D}^2\zeta^\alpha}{ds^2} + \frac{d^\alpha_\sigma}{m_0c^2}\frac{\mathrm{D}\zeta^\sigma}{ds} + \frac{k^\alpha_\sigma}{m_0c^2}\,\zeta^\sigma = -R^{\alpha}_{\cdot\beta\gamma\delta}\left(r^\gamma + \zeta^\gamma\right). \tag{7}$$

This equation is like the equation of forced oscillations, where the curvature tensor is a forcing factor. Weber then finally transformed the equation to

$$\frac{d^2\zeta^\alpha}{dt^2} + \frac{d^\alpha_\sigma}{m_0}\frac{d\zeta^\sigma}{dt} + \frac{k^\alpha_\sigma}{m_0}\,\zeta^\sigma = -c^2R^{\alpha}_{\cdot0\sigma0}\,r^\sigma, \tag{8}$$

which can only be obtained under his assumptions:

1. The length r of the pig to be covariantly constant $r=\sqrt{g_{\mu\nu}r^\mu r^\nu}$, which is a "background" for the infinitesimal displacement of the butt-ends $\zeta^\alpha \ll r^\alpha$ caused by a falling gravitational wave. Note that r isn't the length η of the pig in the "equilibrium state". Weber postulated r^α to be covariantly constant, so r is the "unchanged length". In such a case Weber has actually two detectors at the same time: (1) a pig having the covariantly constant length r, which remains unchanged in the field of a falling gravitational wave, (2) a pig having the length ζ, which, being made from the same material and connected to the first pig, changes its length under the same gravitational wave. In actual experiments a solid-body pig has a monolithic body which reacts as a whole to external influences. In other words, by introducing the splitting term $\eta^\alpha = r^\alpha + \zeta^\alpha$ into the equation of the deviating non-geodesics (2), Weber postulated that a falling gravitational wave is an external entity that forces the particles into resonant oscillations;
2. Because the cylindrical pig is freely suspended, it is in free fall;
3. Christoffel's symbols are all zero, so covariant derivatives became regular derivatives. (Of course, we can choose a specific reference frame where $\Gamma^\alpha_{\mu\nu}=0$ at each given point. Such a reference frame is known as locally geodesic. However, since the curvature tensor is different from zero, $\Gamma^\alpha_{\mu\nu}$ cannot be reduced to zero in a finite area [7]. Therefore, if we connect one particle to a locally geodesic reference frame, in the neighbouring particle $\Gamma^\alpha_{\mu\nu} \neq 0$);
4. The butt-ends of the pig are at rest with respect to the observer ($U^i = 0$) all the time before a gravitational wave passes. This was assumed because the pig was regularly cooled down to a temperature close to 0 K in order to suppress internal molecular motions. With $U^i = 0$, there can only be resonant oscillations of the butt-ends. Parametric oscillations cannot appear there. Therefore Weber and all his followers have expected registration of a signal if a falling gravitational wave produces resonant oscillations in the detector.

Because the same assumptions were applied to the geodesic deviation equation, all that has been said is applicable to a free-mass detector.

Weber didn't solve his final equation (8). He limited himself by using $R^{\alpha}_{\cdot0\sigma0}\,r^\sigma$ as a forcing factor in his calculations of expected oscillations in solid-body detectors. Exact solution of Weber's final equation with all his assumptions was obtained by Borissova in the 1970's [8]. The assumptions actually mean that the solution of the Weber equation (8), with his requirement for r^α and its length $r=\sqrt{g_{\mu\nu}r^\mu r^\nu}$, must be covariantly constant: $\frac{\mathrm{D}r^\alpha}{ds}=0$. Borissova showed that in the case of a gravitational wave linearly polarized

*Weber takes Φ^α as the sum of the returning (elastic) force $k^\alpha_\sigma\zeta^\sigma$ and the force $d^\alpha_\sigma\frac{\mathrm{D}\zeta^\sigma}{ds}$ setting up the damping factor (tensors k^α_σ and d^α_σ describe the peculiarities of the spring).

in the x^2 direction, and propagating along x^1, the equation $\frac{\mathrm{D}r^\alpha}{ds}=0$ gives $r^2=r^2_{(0)}\left[1-A\sin\frac{\omega}{c}\left(ct+x^1\right)\right]$ (the detector oriented along x^2). From this result, she obtained the Weber equation (8) in the form*

$$\frac{d^2\zeta^2}{dt^2}+2\lambda\frac{d\zeta^2}{dt}+\Omega_0^2\,\zeta^2=-A\,\omega^2 r^2_{(0)}\sin\frac{\omega}{c}\left(ct+x^1\right), \quad (9)$$

i. e. an equation of forced oscillations, where the forcing factor is the relative displacement of the particles caused by the gravitational wave. She then obtained the exact solution: the relative displacement $\eta^2=\eta_y$ of the butt-ends is

$$\begin{aligned}\eta^2=r^2_{(0)}\Big[1-A\sin\frac{\omega}{c}\left(ct+x^1\right)\Big]+Me^{-\lambda t}\sin(\Omega t+\alpha)-\\ -\frac{A\,\omega^2 r^2_{(0)}}{(\Omega_0^2-\omega^2)^2}\cos\Big(\omega t+\delta+\frac{\omega}{c}\,x^1\Big),\end{aligned} \quad (10)$$

where $\Omega=\sqrt{\Omega_0^2-\omega^2}$, $\delta=\arctan\frac{2\lambda\omega}{\omega^2-\Omega_0^2}$, while M and α are constants. In this solution the relative oscillations consist of the "basic" harmonic oscillations and relaxing oscillations (first two terms), and the resonant oscillations (third term). As soon as the source's frequency ω coincides with the basic frequency of the detector $\Omega_0=\omega$, resonance occurs: in such a case even weak oscillations may be registered.

Thus, by his equation (6), Weber actually postulated that gravitational waves force rest-particles to undergo relative resonant oscillations. It was amazing that the exact solution showed that! Moreover, his assumptions led to a specific construction of the detectors, where parametric oscillations are obviated. As we show further by the exact solution of the deviation equations, gravitational waves may produce oscillations in only moving particles, in both solid-body and free-mass detectors.

4 Correct solution: a resting detector (Weber's case)

Our solution of the deviation equations depends on a specific formula for the space metric whereby we calculate the Riemann-Christoffel tensor. Because the sources of gravitational waves (double stars, pulsars, etc.) are far away from us, we expect received gravitational waves to be weak and plane. Therefore we consider the well-known metric of weak plane gravitational waves

$$\begin{aligned}ds^2=c^2dt^2-(dx^1)^2-(1+a)(dx^2)^2+\\ +\ 2b\,dx^2dx^3-(1-a)(dx^3)^2,\end{aligned} \quad (11)$$

where a and b are functions of $ct+x^1$ (if propagation is along x^1), while a and b are infinitesimal so that squares and products of their derivatives vanish. The wave field described by this metric has a purely deformational origin, because it is derived from the non-stationarity of the spatial components g_{ik} of the fundamental metric tensor $g_{\alpha\beta}$. This metric is preferred because it satisfies Einstein's equations in vacuum $R_{\alpha\beta}=0$ ($R_{\alpha\beta}$ is Ricci's tensor).

Because we seek solutions applicable to real experiments, we solve the deviation equations in the terms of physically observable quantities†.

The non-geodesic equations of motion (1) have two physically observable projections [11]

$$\begin{aligned}&\frac{dm}{d\tau}-\frac{m}{c^2}\,F_i\mathrm{v}^i+\frac{m}{c^2}\,D_{ik}\mathrm{v}^i\mathrm{v}^k=\frac{\sigma}{c}\,,\\ &\frac{d}{d\tau}(m\mathrm{v}^i)-mF^i+2m\big(D^i_k+A^{\cdot i}_{k\cdot}\big)+m\Delta^i_{kn}\mathrm{v}^k\mathrm{v}^n=f^i,\end{aligned} \quad (12)$$

where m is the relativistic mass of the particle; $\mathrm{v}^i=\frac{dx^i}{d\tau}$ is its three-dimensional observable velocity, the square of which is $\mathrm{v}^2=h_{ik}\mathrm{v}^i\mathrm{v}^k$; $h_{ik}=-g_{ik}+\frac{g_{0i}g_{0k}}{g_{00}}$ is the observable metric tensor; $d\tau=\sqrt{g_{00}}\,dt+\frac{g_{0i}}{c\sqrt{g_{00}}}dx^i$ is the observable time interval, which is different to the coordinate time interval $dt=\frac{1}{c}\,dx^0$; $F_i=\frac{1}{\sqrt{g_{00}}}\left(\frac{\partial\mathrm{w}}{\partial x^i}-\frac{\partial v_i}{\partial t}\right)$ is the observable gravitational inertial force, where w is the gravitational potential, while $\sqrt{g_{00}}=1-\frac{\mathrm{w}}{c^2}$; $v_i=-\frac{c\,g_{0i}}{\sqrt{g_{00}}}$ is the linear velocity of the space rotation; $A_{ik}=\frac{1}{2}\left(\frac{\partial v_k}{\partial x^i}-\frac{\partial v_i}{\partial x^k}\right)+\frac{1}{2c^2}\left(F_iv_k-F_kv_i\right)$ is the tensor of observable angular velocities of the space rotation; $D_{ik}=\frac{1}{2\sqrt{g_{00}}}\frac{\partial h_{ik}}{\partial t}$ the tensor of observable rates of the space deformations; $\Delta^i_{kn}=h^{im}\Delta_{kn,m}$ are the spatially observable Christoffel symbols, built like Christoffel's usual symbols $\Gamma^\alpha_{\mu\nu}=g^{\alpha\sigma}\Gamma_{\mu\nu,\sigma}$ using h_{ik} instead of $g_{\alpha\beta}$; $\sigma=\frac{\Phi_0}{\sqrt{g_{00}}}$ is the observable projection of the non-gravitational force Φ^α onto the observer's time line, while $f^i=\Phi^i$ is its observable projection onto his spatial section.

If a particle rests with respect to an observer ($\mathrm{v}^i=0$), its observable equations of motion (12) take the form

$$\frac{dm_0}{d\tau}=\frac{\sigma}{c}=0\,,\qquad m_0\,F^i=-f^i. \quad (13)$$

Clearly, if a two-particle system is in free fall ($F^i=0$) and also rests with respect to an observer (as happens with a solid-body detector in Weber's experimental statement), a non-gravitational force connecting the particles has no effect on their motion: two resting particles connected by a spring have the same behaviour as free ones.

Therefore, to find what effect is produced by a gravitational wave on a resting solid-body detector or a free-mass detector, we should solve the same Synge equations of the deviating geodesics.

If, as Weber assumed, the observer's reference frame is "synchronous" ($F^i=0$, $A_{ik}=0$, $dt=d\tau$), the metric of weak

*Here $2\lambda=\frac{b}{m_0}$ and $\Omega^2_{(0)}=\frac{k}{m_0}$ are derived from the formula for the non-gravitational force $\Phi^2=-k\zeta^2-b\dot{\zeta}^2$, acting along x^2 in this case. The elastic coefficient of the "spring" is k, the friction coefficient is b.

†Physically observable (chronometrically invariant) are the projections of a four-dimensional quantity onto the time line and the spatial section of an observer [9]. See a brief account of that in [10], for instance.

plane gravitational waves (11) has just $D_{ik} = \frac{1}{2\sqrt{g_{00}}} \frac{\partial h_{ik}}{\partial t} \neq 0$. Let the wave propagate along x^1. Then $D_{22} = -D_{33} = \frac{1}{2}\dot{a}$ and $D_{23} = \frac{1}{2}\dot{b}$, where the dot means differentiation by t. The rest of the components of D_{ik} are zero. In such a case the time observable projection of the Synge equation (4) vanishes, while its spatial observable projection is

$$\frac{d^2\eta^i}{dt^2} + 2D^i_k \frac{d\eta^k}{dt} = 0\,, \tag{14}$$

which is, in component notation,

$$\begin{aligned} &\frac{d^2\eta^1}{dt^2} = 0\,, \\ &\frac{d^2\eta^2}{dt^2} + \frac{da}{dt}\frac{d\eta^2}{dt} + \frac{db}{dt}\frac{d\eta^3}{dt} = 0\,, \\ &\frac{d^2\eta^3}{dt^2} - \frac{da}{dt}\frac{d\eta^3}{dt} + \frac{db}{dt}\frac{d\eta^2}{dt} = 0\,. \end{aligned} \tag{15}$$

The first of these (the deviating acceleration along the wave propagation direction x^1) shows that transverse waves don't produce an effect in the direction of propagation.

We look for exact solutions to the remaining two equations of (15) in the case where a gravitational wave is linearly polarized in the x^2 direction ($b=0$). First integrals of the equations are $\frac{d\eta^2}{dt} = C_1 e^{-a}$ and $\frac{d\eta^3}{dt} = C_2 e^{+a}$. Expanding e^{-a} and e^{+a} into series (high order terms vanish there), we obtain

$$\frac{d\eta^2}{dt} = C_1 (1-a)\,, \qquad \frac{d\eta^3}{dt} = C_2 (1+a)\,. \tag{16}$$

Let the gravitational wave be simple harmonic $\omega = \text{const}$ with a constant amplitude $A = \text{const}$: $a = A \sin \frac{\omega}{c}(ct + x^1)$. We then obtain exact solutions to the equations — the non-zero relative displacements produced in the two-particle system by the gravitational wave falling along x^1:

$$\begin{aligned} \eta^2 &= \dot{\eta}^2_{(0)} \left[t + \frac{A}{\omega}\cos\frac{\omega}{c}(ct + x^1)\right] + \eta^2_{(0)} - \frac{A}{\omega}\dot{\eta}^2_{(0)}\,, \\ \eta^3 &= \dot{\eta}^3_{(0)} \left[t - \frac{A}{\omega}\cos\frac{\omega}{c}(ct + x^1)\right] + \eta^3_{(0)} - \frac{A}{\omega}\dot{\eta}^3_{(0)}\,. \end{aligned} \tag{17}$$

These are the exact solutions of the Synge equation in a particular case, realised today in all solid-body and free-mass detectors. Looking at the solutions, we conclude:

> Transverse gravitational waves of a deformational sort may produce an effect in a two-particle system, resting as a whole with respect to the observer, only if the particles initially oscillate with respect to each other. If the particles are at rest in the initial moment of time, a falling gravitational wave cannot produce relative displacement of the particles.

Therefore the correct theory of a gravitational wave detector we have built states:

> Solid-body and free-mass detectors of current construction cannot register gravitational waves in principle; in cooling a solid-body detector and initially placing two distant mirrors at rest in a free-mass detector, inherent free oscillations are suppressed, thereby preventing registration of gravitational waves by the detectors.

In order to make the detectors sensitive to gravitational waves, we propose the following changes to their current construction:

For a free-mass detector: Introduce relative oscillations of the mirrors along their mutual line of sight. Such a modified system may have a reaction to a falling gravitational wave as an add-on to the relative velocity of the mirrors on the background of their basic relative oscillations.

For a solid-body detector: Don't cool the cylindrical pig, or better, apply relative oscillations of the butt-ends. Then the pig may have a reaction to a falling gravitational wave: an add-on to the noise of the self-deforming oscillations regularly detected as a piezoelectric effect*.

By the foregoing modifications to the exact theory of a gravitational wave detector, a solid-body detector, and especially a free-mass detector, may register gravitational waves.

Our theoretical result shows that to detect gravitational waves, the best method would be a detector consisting of two moving "particles". From the purely theoretical perspective, this is a general case of the deviation equations, where both particles move with respect to the observer at the initial moment of time. We obtain therefore, exact solutions for the general case and, as a result, consider detectors built on moving "particles" — a suspended, self-vibrating solid-body pig or suspended, vibrating mirrors in a free-mass detector.

5 Correct solution: a moving detector (general case)

If Weber had solved the deviation equation in conjunction with the equations of motion, he would have come to the same conclusion as us: gravitational waves of the deformational sort may produce an effect in a two-particle system only if the particles are in motion. Therefore we are going to solve the deviation equation in conjunction with the equations of motion in the general case where both particles move initially

*Because of this, it is most probable that Weber really detected gravitational waves in his experiments of 1968–1970 [3, 4, 5] where he used room-temperature detectors ". . . spaced about 2 km. A number of coincident events have been observed, with extremely small probability that they are statistical. It is clear that on rare occasions these instruments respond to a common external excitation which may be gravitational radiation" [3]. "Coincidences have been observed on gravitational-radiation detectors over a base line of about 1000 km at Argonne National Laboratory and at the University of Maryland. The probability that all of these coincidences were accidental is incredibly small" [4]. "Other experiments involve observations to rule out the possibility that the detectors are being excited electromagnetically. These results are evidence supporting an earlier claim that gravitational radiation is being observed" [5].

We both highly appreciate the work of Joseph Weber (1919–2000). Surely, if he was still alive he would be enthusiastic about our current results, and with us, immediately undertake new experiments for the detection of gravitational waves.

with respect to the observer ($U^i \neq 0$). (We mean that both particles move at the same velocity.)

We do this with the Synge-Weber equation of the deviating non-geodesics, because the Synge equation of the deviating geodesics is actually the same when the right side is zero.

We write the Synge-Weber equation (2) in the expanded form (with similar terms reduced)

$$\frac{d^2\eta^\alpha}{ds^2}+2\Gamma^\alpha_{\mu\nu}\frac{d\eta^\mu}{ds}U^\nu+\frac{\partial\Gamma^\alpha_{\beta\delta}}{\partial x^\gamma}U^\beta U^\delta\eta^\gamma=\frac{1}{m_0c^2}\frac{\partial\Phi^\alpha}{\partial x^\gamma}\eta^\gamma, \quad (18)$$

where ds may be expressed through the observable time interval $d\tau=\sqrt{g_{00}}\,dt+\frac{g_{0i}}{c\sqrt{g_{00}}}dx^i$ as $ds=c\,d\tau\sqrt{1-\mathrm{v}^2/c^2}$.

According to Zelmanov [9], any vector Q^α has two observable projections $\frac{Q_0}{\sqrt{g_{00}}}$ and Q^i, where the time projection may be calculated as $\frac{Q_0}{\sqrt{g_{00}}}=\sqrt{g_{00}}\,Q^0-\frac{1}{c}\,v_iQ^i$. We denote $\sigma=\frac{\Phi_0}{\sqrt{g_{00}}}$ and $f^i=\Phi^i$ for the connecting force Φ^α, while $\varphi=\frac{\eta_0}{\sqrt{g_{00}}}$ and η^i for the deviation η^α.

We consider the Synge-Weber equation (18) in a non-relativistic case, because the velocity of the particles is obviously small. In such a case, in the metric of weak plane gravitational waves (11), we have*

$$\begin{gathered} d\tau=dt\,, \qquad \eta^0=\eta_0=\varphi\,, \quad \Phi^0=\Phi_0=\sigma\,, \\ \Gamma^0_{kn}=\frac{1}{c}D_{kn}\,, \quad \Gamma^i_{0k}=\frac{1}{c}D^i_k\,, \quad \Gamma^i_{kn}=\Delta^i_{kn}\,, \end{gathered} \quad (19)$$

while all other Christoffel symbols are zero. We obtain the time and spatial observable projections of the Synge-Weber equation (18), which are

$$\begin{gathered} \frac{d^2\varphi}{dt^2}+\frac{2}{c}D_{kn}\frac{d\eta^k}{dt}\mathrm{v}^n+\left(\varphi\frac{\partial D_{kn}}{\partial t}+c\frac{\partial D_{kn}}{\partial x^m}\eta^m\right)\frac{\mathrm{v}^k\mathrm{v}^n}{c^2}= \\ =\frac{1}{m_0}\left(\frac{\varphi}{c}\frac{\partial\sigma}{\partial t}+\frac{\partial\sigma}{\partial x^m}\eta^m\right), \\ \frac{d^2\eta^i}{dt^2}+\frac{2}{c}D^i_k\left(\frac{d\varphi}{dt}\mathrm{v}^k+c\frac{d\eta^k}{dt}\right)+2\Delta^i_{kn}\frac{d\eta^k}{dt}\mathrm{v}^n+ \\ +2\left(\frac{\varphi}{c}\frac{\partial D^i_k}{\partial t}+\frac{\partial D^i_k}{\partial x^m}\eta^m\right)\mathrm{v}^k+\left(\frac{\varphi}{c}\frac{\partial\Delta^i_{kn}}{\partial t}+\frac{\partial\Delta^i_{kn}}{\partial x^m}\eta^m\right)\mathrm{v}^k\mathrm{v}^n= \\ =\frac{1}{m_0}\left(\frac{\varphi}{c}\frac{\partial f^i}{\partial t}+\frac{\partial f^i}{\partial x^m}\eta^m\right). \end{gathered} \quad (20)$$

We solve the deviation equations (20) in the field of a weak plane gravitational wave falling along x^1 and linearly polarized in the x^2 direction ($b=0$). In such a field we have

$$\begin{gathered} D_{22}=-D_{33}=\frac{1}{2}\,\dot{a}\,, \qquad \frac{d}{dx^1}=\frac{1}{c}\frac{d}{dt}\,, \\ \Delta^1_{22}=-\Delta^1_{33}=-\frac{1}{2c}\,\dot{a}\,, \quad \Delta^2_{12}=-\Delta^3_{13}=\frac{1}{2c}\,\dot{a}\,, \end{gathered} \quad (21)$$

so that the deviation equations (20) in component form are

*By the metric of weak plane gravitational waves (11), there is no difference between upper and lower indices.

$$\begin{gathered} \frac{d^2\varphi}{dt^2}+\frac{\dot{a}}{c}\left(\frac{d\eta^2}{dt}\mathrm{v}^2-\frac{d\eta^3}{dt}\mathrm{v}^3\right)+ \\ +\frac{\ddot{a}}{2c^2}\left(\varphi+\eta^1\right)\left((\mathrm{v}^2)^2-(\mathrm{v}^3)^2\right)=\frac{1}{m_0}\left(\frac{1}{c}\frac{\partial\sigma}{\partial t}+\frac{\partial\sigma}{\partial x^m}\eta^m\right), \\ \frac{d^2\eta^1}{dt^2}-\frac{\dot{a}}{c}\left(\frac{d\eta^2}{dt}\mathrm{v}^2-\frac{d\eta^3}{dt}\mathrm{v}^3\right)- \\ -\frac{\ddot{a}}{2c^2}\left(\varphi+\eta^1\right)\left((\mathrm{v}^2)^2-(\mathrm{v}^3)^2\right)=\frac{1}{m_0}\left(\frac{1}{c}\frac{\partial f^1}{\partial t}+\frac{\partial f^1}{\partial x^m}\eta^m\right), \\ \frac{d^2\eta^2}{dt^2}+\frac{\dot{a}}{c}\left(\frac{d\varphi}{dt}+\frac{d\eta^1}{dt}\right)\mathrm{v}^2+\dot{a}\,\frac{d\eta^2}{dt}\left(1+\frac{\mathrm{v}^1}{c}\right)+ \\ +\frac{\ddot{a}}{c}\left(\varphi+\eta^1\right)\left(1+\frac{\mathrm{v}^1}{c}\right)\mathrm{v}^2=\frac{1}{m_0}\left(\frac{1}{c}\frac{\partial f^2}{\partial t}+\frac{\partial f^2}{\partial x^m}\eta^m\right), \\ \frac{d^2\eta^3}{dt^2}-\frac{\dot{a}}{c}\left(\frac{d\varphi}{dt}+\frac{d\eta^1}{dt}\right)\mathrm{v}^3-\dot{a}\,\frac{d\eta^3}{dt}\left(1+\frac{\mathrm{v}^1}{c}\right)- \\ -\frac{\ddot{a}}{c}\left(\varphi+\eta^1\right)\left(1+\frac{\mathrm{v}^1}{c}\right)\mathrm{v}^3=\frac{1}{m_0}\left(\frac{1}{c}\frac{\partial f^3}{\partial t}+\frac{\partial f^3}{\partial x^m}\eta^m\right). \end{gathered} \quad (22)$$

This is a system of 2nd order differential equations with respect to φ, η^1, η^2, η^3, where the variable coefficients of the functions are the quantities $\dot{a}$, $\ddot{a}$, v^1, v^2, v^3.

We may find a from the given metric of the gravitational wave field, while v^i are the solutions to the non-geodesic equations of motion (12). By the given non-relativistic case in a field of weak plane linearly polarized gravitational wave, the equations of motion take the form

$$\begin{gathered} \frac{\dot{a}}{2c}\left((\mathrm{v}^2)^2-(\mathrm{v}^3)^2\right)=\frac{\sigma}{m_0}\,, \\ \frac{d\mathrm{v}^1}{dt}-\frac{\dot{a}}{2c}\left((\mathrm{v}^2)^2-(\mathrm{v}^3)^2\right)=\frac{f^1}{m_0}\,, \\ \frac{d\mathrm{v}^2}{dt}+\dot{a}\,\mathrm{v}^2\left(1+\frac{\mathrm{v}^1}{c}\right)=\frac{f^2}{m_0}\,, \\ \frac{d\mathrm{v}^3}{dt}-\dot{a}\,\mathrm{v}^3\left(1+\frac{\mathrm{v}^1}{c}\right)=\frac{f^3}{m_0}\,. \end{gathered} \quad (23)$$

5.1 Solution for a free-mass detector

We first find the solution for a simple case, where two particles don't interact with each other ($\Phi^\alpha=0$) — the right side is zero in the equations. This is a case of a free-mass detector. We find the quantities v^i from the equations of motion (23), which, since $\Phi^\alpha=0$, become geodesic

$$\begin{gathered} (\mathrm{v}^2)^2-(\mathrm{v}^3)^2=0\,, \qquad \frac{d\mathrm{v}^1}{dt}=0\,, \\ \frac{d\mathrm{v}^2}{dt}+\dot{a}\,\mathrm{v}^2=0\,, \qquad \frac{d\mathrm{v}^3}{dt}+\dot{a}\,\mathrm{v}^3=0\,. \end{gathered} \quad (24)$$

From this we see that a transverse gravitational wave doesn't produce an effect in the longitudinal direction: $\mathrm{v}^1=\mathrm{v}^1_{(0)}=\mathrm{const}$. Therefore, henceforth, $\mathrm{v}^1_{(0)}=0$.

The remaining equations of (24) may be integrated without problems. We obtain: $\mathrm{v}^2 = \mathrm{v}^2_{(0)} e^{-a}$, $\mathrm{v}^3 = \mathrm{v}^3_{(0)} e^{+a}$. Assuming the wave simple harmonic, $\omega = \text{const}$, with a constant amplitude, $A = \text{const}$, i. e. $a = A \sin \frac{\omega}{c}(ct + x^1)$, and expanding the exponent into series, we obtain

$$\begin{aligned} \mathrm{v}^2 &= \mathrm{v}^2_{(0)} \left[1 - A \sin \frac{\omega}{c}(ct + x^1)\right], \\ \mathrm{v}^3 &= \mathrm{v}^3_{(0)} \left[1 + A \sin \frac{\omega}{c}(ct + x^1)\right], \end{aligned} \tag{25}$$

i. e. a gravitational wave has an effect only in directions orthogonal to its propagation. Clearly, a gravitational wave doesn't affect particles at rest with respect to the local space where the wave propagates.

Substituting the solutions (25) into the equations of the deviating non-geodesics (22) and setting the right side to zero as for geodesics, we obtain

$$\begin{aligned} &\frac{d^2\varphi}{dt^2} + \frac{\dot a}{c}\left(\frac{d\eta^2}{dt}\mathrm{v}^2_{(0)} - \frac{d\eta^3}{dt}\mathrm{v}^3_{(0)}\right) = 0, \\ &\frac{d^2\eta^1}{dt^2} - \frac{\dot a}{c}\left(\frac{d\eta^2}{dt}\mathrm{v}^2_{(0)} - \frac{d\eta^3}{dt}\mathrm{v}^3_{(0)}\right) = 0, \\ &\frac{d^2\eta^2}{dt^2} + \dot a\frac{d\eta^2}{dt} + \frac{\dot a}{c}\left(\frac{d\varphi}{dt} + \frac{d\eta^1}{dt}\right)\mathrm{v}^2_{(0)} + \frac{\ddot a}{c}\left(\varphi + \eta^1\right)\mathrm{v}^2_{(0)} = 0, \\ &\frac{d^2\eta^3}{dt^2} - \dot a\frac{d\eta^3}{dt} - \frac{\dot a}{c}\left(\frac{d\varphi}{dt} + \frac{d\eta^1}{dt}\right)\mathrm{v}^2_{(0)} - \frac{\ddot a}{c}\left(\varphi + \eta^1\right)\mathrm{v}^2_{(0)} = 0. \end{aligned} \tag{26}$$

Summing the first two equations and integrating the sum, we obtain $\varphi + \eta^1 = B_1 t + B_2$, where $B_{1,2}$ are integration constants. Substituting these into the other two, we obtain

$$\begin{aligned} &\frac{d^2\eta^2}{dt^2} + \dot a\frac{d\eta^2}{dt} + \frac{\dot a}{c}B_1\mathrm{v}^2_{(0)} + \frac{\ddot a}{c}\left(B_1 t + B_2\right)\mathrm{v}^2_{(0)} = 0, \\ &\frac{d^2\eta^3}{dt^2} - \dot a\frac{d\eta^2}{dt} - \frac{\dot a}{c}B_1\mathrm{v}^3_{(0)} - \frac{\ddot a}{c}\left(B_1 t + B_2\right)\mathrm{v}^3_{(0)} = 0. \end{aligned} \tag{27}$$

The equations differ solely in the sign of a, and can therefore be solved in the same way. We introduce a new variable $y = \frac{d\eta^2}{dt}$. Then we have a linear uniform equation of the 1st order with respect to y

$$\dot y + \dot a y = -\frac{\dot a}{c}B_1\mathrm{v}^2_{(0)} - \frac{\ddot a}{c}\left(B_1 t + B_2\right)\mathrm{v}^2_{(0)}, \tag{28}$$

which has the solution

$$y = e^{-F}\left(y_0 + \int_0^t g(t)\, e^F dt\right), \quad F(t) = \int_0^t f(t)\, dt, \tag{29}$$

where $F(t) = \dot a$, $g(t) = -\frac{\dot a}{c}B_1\mathrm{v}^2_{(0)} - (B_1 t + B_2)\mathrm{v}^2_{(0)}$. Expanding the exponent into series in the solution, and then integrating, we obtain

$$\begin{aligned} y = \dot\eta^2 &= \dot\eta^2_{(0)}\left[1 - A\sin\frac{\omega}{c}(ct + x^1)\right] - \\ &- \frac{A\omega}{c}\mathrm{v}^2_{(0)}\left(B_1 t + B_2\right)\cos\frac{\omega}{c}(ct + x^1) + \frac{A\omega}{c}B_2\mathrm{v}^2_{(0)}. \end{aligned} \tag{30}$$

Integrating this equation, and applying the same method for η^3, we arrive at the final solutions: the relative displacements η^2 and η^3 in a free-mass detector are

$$\begin{aligned} \eta^2 &= \eta^2_{(0)} + \left(\dot\eta^2_{(0)} + \frac{A\omega B_2\mathrm{v}^2_{(0)}}{c}\right)t + \frac{A}{\omega}\left(\dot\eta^2_{(0)} - \frac{\mathrm{v}^2_{(0)}}{c}B_1\right)\times \\ &\times\left[\cos\frac{\omega}{c}(ct + x^1) - 1\right] - \frac{A\mathrm{v}^2_{(0)}}{c}\left(B_1 t + B_2\right)\sin\frac{\omega}{c}(ct + x^1), \end{aligned} \tag{31}$$

$$\begin{aligned} \eta^3 &= \eta^3_{(0)} + \left(\dot\eta^3_{(0)} - \frac{A\omega B_2\mathrm{v}^3_{(0)}}{c}\right)t - \frac{A}{\omega}\left(\dot\eta^3_{(0)} - \frac{\mathrm{v}^3_{(0)}}{c}B_1\right)\times \\ &\times\left[\cos\frac{\omega}{c}(ct + x^1) - 1\right] + \frac{A\mathrm{v}^3_{(0)}}{c}\left(B_1 t + B_2\right)\sin\frac{\omega}{c}(ct + x^1). \end{aligned} \tag{32}$$

With $\dot\eta^2$ and $\dot\eta^3$, we integrate the first two equations of (26). We obtain thereby the relative displacement η^1 in a free-mass detector and the time shift φ at its ends, thus

$$\eta^1 = \dot\eta^1_{(0)}t - \frac{A}{\omega c}\left(\mathrm{v}^2_{(0)}\dot\eta^2_{(0)} - \mathrm{v}^3_{(0)}\dot\eta^3_{(0)}\right)\left[1 - \cos\frac{\omega}{c}(ct + x^1)\right] + \eta^1_{(0)}, \tag{33}$$

$$\varphi = \dot\varphi_{(0)}t + \frac{A}{\omega c}\left(\mathrm{v}^2_{(0)}\dot\eta^2_{(0)} - \mathrm{v}^3_{(0)}\dot\eta^3_{(0)}\right)\left[1 - \cos\frac{\omega}{c}(ct + x^1)\right] + \eta^1_{(0)}. \tag{34}$$

Finally, we substitute φ and η^1 into $\varphi + \eta^1 = B_1 t + B_2$ to fix the integration constants $B_1 = \dot\varphi_{(0)} + \dot\eta^1_{(0)}$ and $B_2 = \varphi_{(0)} + \eta^1_{(0)}$.

Thus we have obtained the solutions to the Synge equation of deviating geodesics. We see that relative displacements of two free particles in the directions x^2 and x^3, transverse to that of gravitational wave propagation consist of:

1. Displacements, increasing linearly with time;
2. Harmonic oscillations at the frequency ω of a falling gravitational wave;
3. Oscillations, the amplitude of which increases linearly with time (last term in the solutions).

The first two of the displacements are permitted in the transverse direction x^2 or x^3, only if the particles initially move in this direction with respect to the local space ($\mathrm{v}^2 \neq 0$ or $\mathrm{v}^3 \neq 0$) or with respect to each other ($\dot\eta^2 \neq 0$ or $\dot\eta^3 \neq 0$). For instance, if they are at rest with respect to x^2, an x^1-directed gravitational wave doesn't displace them in this direction.

The third of the displacements is permitted only if the particles initially move with respect to each other in the longitudinal direction ($\dot\eta^1 \neq 0$).

We see from the solution for η^1 that gravitational waves may displace the particles even in the same direction of the wave propagation, if the particles initially move in this direction with respect to each other.

The solution φ is the time shift in the clocks located at both particles, caused by a falling gravitational wave*. From (34), this effect is permitted if the particles move both with

*We assume $\varphi_{(0)} = 0$: time count starts from zero. We assume as well $\dot\varphi_{(0)} = 0$: time flows uniformly in the absence of a wave gravitational field.

respect to the local space and each other in at least one of the transverse directions x^2 and x^3.

In view of these results, we propose a new experimental statement for the detection of gravitational waves, based on a free-mass detector.

New experiment for a free-mass detector: A free-mass detector, where two mirrors are suspended and vibrating so that they have free oscillations with respect to each other or along parallel (vertical or horizontal) lines. With the mirrors oscillating along parallel lines, such a system moves with respect to the local space ($\mathrm{v}^i_{(0)} \neq 0$), while with the mirrors oscillating with respect to each other the system has non-stationary relative displacements of the butt-ends ($\eta^i_{(0)} \neq 0$, $\dot{\eta}^i_{(0)} \neq 0$). According to the exact theory of a free-mass detector given above, a falling gravitational wave produces a relative displacement of the mirrors, that may be registered with a laser range-finder (or similar system). Moreover, as the theory predicts, a time shift is produced in the mirrors, that may be registered by synchronized clocks located with each of the mirrors: their asynchronization implies a gravitational wave detection.

5.2 Solution for a solid-body detector

We assume an elastic force connecting two particles in a solid-body detector to be $\Phi^\alpha = -k^\alpha_\sigma x^\sigma$, where k^α_σ is the elastic coefficient. We assume the force Φ^α to be independent of time, i. e. $k^0_\sigma = 0$. In such a case the equations of motion of the particles (23) take the form

$$\left.\begin{aligned} &(\mathrm{v}^2)^2 - (\mathrm{v}^3)^2 = 0\,, \\ &\frac{d\mathrm{v}^1}{dt} - \frac{\dot{a}}{2c}\Big((\mathrm{v}^2)^2 - (\mathrm{v}^3)^2\Big) = -\frac{k^1_\sigma}{m_0}\,x^\sigma, \\ &\frac{d\mathrm{v}^2}{dt} + \dot{a}\,\mathrm{v}^2\left(1 + \frac{\mathrm{v}^1}{c}\right) = -\frac{k^2_\sigma}{m_0}\,x^\sigma, \\ &\frac{d\mathrm{v}^3}{dt} - \dot{a}\,\mathrm{v}^3\left(1 + \frac{\mathrm{v}^1}{c}\right) = -\frac{k^3_\sigma}{m_0}\,x^\sigma. \end{aligned}\right. \tag{35}$$

Thus a transverse gravitational wave doesn't produce an effect in the longitudinal direction x^1: $\mathrm{v}^1 = \mathrm{v}^1_{(0)} = \mathrm{const}$. Therefore, henceforth, $\mathrm{v}^1 = 0$ and $k^1_\sigma = 0$. In such a case the equations of motion take the form

$$\frac{d\mathrm{v}^2}{dt} + \dot{a}\,\mathrm{v}^2 = -\frac{k^2_\sigma}{m_0}\,x^\sigma, \qquad \frac{d\mathrm{v}^3}{dt} - \dot{a}\,\mathrm{v}^3 = -\frac{k^3_\sigma}{m_0}\,x^\sigma. \tag{36}$$

The equations differ solely by the sign of $\dot{a}$. Therefore we solve only the first of them. The second equation may be solved following the same method.

Let $k^2_\sigma = k^3_\sigma = k = \mathrm{const}$, i. e. the solid-body pig is elastic in only two directions transverse to the direction x^1 of the gravitational wave propagation. With that, the equation of motion in the x^2 direction is

$$\frac{d^2x^2}{dt^2} + \frac{k}{m_0}\,x^2 = -A\,\omega\cos\frac{\omega}{c}\left(ct + x^1\right)\frac{dx^2}{dt}\,. \tag{37}$$

Denoting $x^2 \equiv x$, $\frac{k}{m_0} = \Omega^2$, $A\omega = -\mu$, we reduce this equation to the form

$$\ddot{x} + \Omega^2 x = \mu\cos\frac{\omega}{c}\left(ct + x^1\right)\dot{x}\,, \tag{38}$$

where μ is the so-called "small parameter". This is a "quasi-harmonic" equation: with $\mu = 0$, such an equation is a harmonic oscillation equation; while if $\mu \neq 0$ the right side plays the rôle of an forcing factor — we obtain a forced oscillation equation.

We solve this equation using the small parameter method of Poincaré, known also as the perturbation method: we consider the right side as a perturbation of a harmonic oscillation described by the left side. The Poincaré method is related to exact solution methods, because a solution produced with the method is a power series expanded by the small parameter μ (see Lefschetz, Chapter XII, §2 [12]).

Before we solve (38) we introduce a new variable $t' = \Omega t$ in order to make it dimensionless as in [12], and $\mu' = \frac{\mu}{\Omega}$

$$\ddot{x} + x = \mu'\cos\frac{\omega}{\Omega c}\left(ct' + \Omega x^1\right)\dot{x}\,, \tag{39}$$

where we differentiate by t'. A general solution of this equation, representable as the equivalent system

$$\dot{x} = y\,, \qquad \dot{y} = -x + \mu'\cos\frac{\omega}{\Omega c}\left(ct' + \Omega x^1\right)y \tag{40}$$

with the initial data $x_{(0)}$ and $y_{(0)}$ at $t' = 0$, is determined by the series pair (Lefschetz)

$$\left.\begin{aligned} x &= P_0\left(x_{(0)}, y_{(0)}, t'\right) + \mu' P_1\left(x_{(0)}, y_{(0)}, t'\right) + \ldots \\ y &= \dot{P}_0\left(x_{(0)}, y_{(0)}, t'\right) + \mu' \dot{P}_1\left(x_{(0)}, y_{(0)}, t'\right) + \ldots \end{aligned}\right\}. \tag{41}$$

We substitute these into (40) and, equating coefficients in the same orders of μ', obtain the recurrent system

$$\left.\begin{aligned} &\ddot{P}_0 + P_0 = 0 \\ &\ddot{P}_1 + P_1 = \dot{P}_0\cos\frac{\omega}{\Omega c}\left(ct' + \Omega x^1\right) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{42}$$

with the initial data $P_0(0) = \xi$, $\dot{P}_0(0) = \vartheta$, $P_1(0) = \dot{P}_1(0) = 0$ $(n > 0)$ at $t' = 0$. Because the amplitude A (we have it in the variable $\mu' = -\frac{\omega}{\Omega}A$) is small, this problem takes only the first two equations into account. The first of them is a harmonic oscillation equation, with the solution

$$P_0 = \xi\cos t' + \vartheta\sin t', \tag{43}$$

while the second equation, with this solution, is

$$\ddot{P}_1 + P_1 = \left(-\xi\sin t' + \vartheta\cos t'\right)\cos\frac{\omega}{\Omega c}\left(ct' + \Omega x^1\right). \tag{44}$$

This is a linear uniform equation. We solve it following Kamke (Part III, Chapter II, §2.5 in [13]). The solution is*

*Here we go back to the initial variables.

$$P_1 = \frac{\vartheta\Omega^2}{2}\left\{\frac{\cos\left[(\Omega-\omega)t-\frac{\omega}{c}x^1\right]}{\Omega^2-(\Omega-\omega)^2}+\frac{\cos\left[(\Omega+\omega)t+\frac{\omega}{c}x^1\right]}{\Omega^2-(\Omega+\omega)^2}\right\} - \\ -\frac{i\xi\Omega^2}{2}\left\{\frac{\sin\left[(\Omega-\omega)t-\frac{\omega}{c}x^1\right]}{\Omega^2-(\Omega-\omega)^2}+\frac{\sin\left[(\Omega+\omega)t+\frac{\omega}{c}x^1\right]}{\Omega^2-(\Omega+\omega)^2}\right\}, \tag{45}$$

where the brackets contain the real and imaginary parts of the formula $e^{i(\Omega-\omega)t-\frac{\omega}{c}x^1}+e^{i(\Omega+\omega)t+\frac{\omega}{c}x^1}$. Going back to $x^2=x$, we obtain the final solution in the reals

$$x^2 = \xi\cos\Omega t+\vartheta\sin\Omega t - \\ -\frac{A\omega\Omega\vartheta}{2}\left\{\frac{\cos\left[(\Omega-\omega)t-\frac{\omega}{c}x^1\right]}{\Omega^2-(\Omega-\omega)^2}+\frac{\cos\left[(\Omega+\omega)t+\frac{\omega}{c}x^1\right]}{\Omega^2-(\Omega+\omega)^2}\right\}, \tag{46}$$

while the solution for x^3 will differ solely in the sign of the amplitude A.

With this result we solve the equations of the deviating non-geodesics (22). Because a solid-body detector has a freedom for motion less than a free-mass detector, we assume $\mathrm{v}^1=0$, $\mathrm{v}^2=\mathrm{v}^3$, $\Phi^1=0$, $\Phi^2=-\frac{k}{m_0}\eta^2$, $\Phi^3=-\frac{k}{m_0}\eta^3$. Note that $\mathrm{v}^2=\mathrm{v}^3$ means that the initial conditions ξ and ϑ are the same in both the directions x^2 and x^3. Therefore we obtain

$$\frac{d^2\varphi}{dt^2}=0\,, \qquad \frac{d^2\eta^1}{dt^2}=0\,, \tag{47}$$

i. e. a gravitational wave doesn't change both the vertical size of the pig and the time shift φ at its butt-ends: we may put $\varphi=0$ and $\eta^1=0$. With all these, the deviation equation along x^2 takes the form*

$$\frac{d^2\eta^2}{dt^2}+\frac{k}{m_0}\eta^2=-A\,\omega\cos\frac{\omega}{c}\left(ct+x^1\right)\frac{d\eta^2}{dt}\,, \tag{48}$$

having the same form as equation (37). So the solution η^2 is like (46), but with the difference that the initial constants ξ and ϑ depend on $\eta^2_{(0)}$, $\eta^3_{(0)}$ and $\dot{\eta}^2_{(0)}$, $\dot{\eta}^3_{(0)}$. It is

$$\eta^2 = \xi\cos\Omega t+\vartheta\sin\Omega t - \\ -\frac{A\omega\Omega\vartheta}{2}\left\{\frac{\cos\left[(\Omega-\omega)t-\frac{\omega}{c}x^1\right]}{\Omega^2-(\Omega-\omega)^2}+\frac{\cos\left[(\Omega+\omega)t+\frac{\omega}{c}x^1\right]}{\Omega^2-(\Omega+\omega)^2}\right\}. \tag{49}$$

Thus two spring-connected particles in the field of a gravitational wave may experience the following effects:

1. Free relative oscillations at a frequency Ω;
2. Forced relative oscillations, caused by the gravitational wave of frequency ω; they occur in the directions transverse to the wave propagation;
3. Resonant oscillations, which occur as soon as the gravitational wave's frequency becomes double the frequency of the particle's free oscillation ($\omega=2\Omega$); in such a case even weak oscillations caused by the gravitational wave may be detected;

*We write and solve only the equation for η^2, because it differs to that for η^3 solely by the sign of the amplitude A. See (22).

The second and third effects are permitted only if the particles have an initial relative oscillation. If there is no initial oscillation, gravitational waves cannot produce an effect in such a system. Owing to this result, we propose a new experimental statement for the detection of gravitational waves by a solid-body detector.

New experiment for a solid-body detector: Use a solid-body cylindrical pig, horizontally suspended and self-vibrating so that there are relative oscillations of its butt-ends ($\eta^2_{(0)}\neq 0$, $\dot{\eta}^2_{(0)}\neq 0$). Such an oscillation may be induced by alternating electromagnetic current or something like this. Or, alternatively, use a similarly suspended, vibrating pig so that it has an oscillation in the horizontal plane. Such a system has a non-zero velocity with respect to the observer's local space ($\mathrm{v}^2_{(0)}\neq 0$, $\mathrm{v}^3_{(0)}\neq 0$). Both systems, according to the exact theory of a solid-body detector, may have a reaction to gravitational waves (up to resonance) that may be measured as a piezo-effect in the pig.

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