A Possible General Approach Regarding the Conformability of Angular Observables with the Mathematical Rules of Quantum Mechanics

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The conformability of angular observales (angular momentum and azimuthal angle) with the mathematical rules of quantum mechanics is a question which still rouses debates. It is valued negatively within the existing approaches which are restricted by two amendable presumptions. If the respective presumptions are removed one can obtain a general approach in which the mentioned question is valued positively.

1 Introduction

In the last decades the pair of angular observables $L_z - \varphi$ (angular momentum — azimuthal angle) was and still is regarded as being unconformable to the accepted mathematical rules of Quantum Mechanics (QM) (see [1–24]). The unconformity is identified with the fact that , in some cases of circular motions, for the respective pair the Robertson-Schrödinger uncertainty relation (RSUR) is not directly applicable. That fact roused many debates and motivated various approaches planned to elucidate in an acceptable manner the missing conformability. But so far such an elucidation was not ratified (or admited unanimously) in the scientific literature.

A minute inspection of the things shows that in the main all the alluded approaches have a restricted character due to the presumptions (***P***):

- $\boldsymbol{P}_1$: Consideration of RSUR as a twofold reference element by: (i) proscription of its direct $L_z - \varphi$ descendant, and (ii) substitution of the respective descendant with some RSUR-mimic relations;
- $\boldsymbol{P}_2$: Discussion only of the systems with sharp circular rotations (SCR).

But the mentioned presumptions are amendable because they conflict with the following facts (***F***):

- $\boldsymbol{F}_1$: Mathematically, the RSUR is only a secondary piece, of limited validity, resulting from a generally valid element represented by a Cauchy Schwarz formula (CSF) (see down Section 4);
- $\boldsymbol{F}_2$: From a natural physical viewpoint the $L_z - \varphi$ pair must be considered in connection not only with SCR but also with any orbital (spatial) motions (e.g. with the non-circular rotations (NCR), presented below in Section 3).

The above facts suggest that for the $L_z - \varphi$ problem ought to search new approaches, by removing the mentioned premises $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. As we know until now such approaches were not promoted in the publications from the main stream of scientific literature. In this paper we propose a possible general approach of the mentioned kind, able to ensure a natural conformability of the $L_z - \varphi$ pair with the prime mathematical rules of QM.

For distiguinshing our proposal from the alluded restricted approaches, in the next Section we present briefly the respective approaches, including their main assertions and a set of unavoidable shortcomings which trouble them destructively. Then, in Section 3, we disclose the existence of two examples of NCR which are in discordance with the same approaches.

The alluded shorcomings and discordances reenforce the interest for new and differently oriented approaches of the $L_z - \varphi$ problem. Such an approach, of general perspective, is argued and detailed below in our Section 4. We end the paper in Section 5 with some associate conclusions.

2 Briefly on the restricted approaches

Certainly, for the history of the $L_z - \varphi$ problem, the first reference element was the Robertson Schrödinger uncertainty relation (RSUR) introduced [25, 26] within the mathematical formalism of QM. In terms of usual notations from QM the RSUR is written as

$$\Delta_\psi A \cdot \Delta_\psi B \geqslant \frac{1}{2} \left| \left\langle \left[\hat{A}, \hat{B} \right] \right\rangle_\psi \right| , \tag{1}$$

where $\Delta_\psi A$ and $\langle (\ldots) \rangle_\psi$ signify the standard deviation of the observable A respectively the mean value of $(\ldots)$ in the state described by the wave function ψ, while $\left[\hat{A}, \hat{B} \right]$ denote the commutator of the operators $\hat{A}$ and $\hat{B}$ (for more details about the notations and validity regarding the RSUR 1, see the next Section).

The attempts for application of RSUR (1) to the case with $A = L_z$ and $B = \varphi$, i.e. to the $L_z - \varphi$ pair, evidenced the following intriguing facts.

On the one hand, according to the usual procedures of QM [27], the observables L_z and φ should be described by the conjugated operators

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi} , \qquad \hat{\varphi} = \varphi \cdot \tag{2}$$

respectively by the commutation relation

$$\left[\hat{L}_z, \hat{\varphi}\right] = -i\hbar \,. \tag{3}$$

So for the alluded pair the RSUR (1) requires for its direct descendant the relation

$$\Delta_\psi L_z \cdot \Delta_\psi \varphi \geqslant \frac{\hbar}{2} \,. \tag{4}$$

On the other hand this last relation is explicitly inapplicable in cases of angular states regarding the systems with sharp circular rotations (SCR). The respective inapplicability is pointed out here bellow.

As examples with SCR can be quoted : (i) a particle (bead) on a circle, (ii) a 1D (or fixedaxis) rotator and (iii) non-degenerate spatial rotations. One finds examples of systems with spatial rotations in the cases of a particle on a sphere, of 2D or 3D rotators and of an electron in a hydrogen atom respectively. The mentioned rotations are considered as non-degenerate if all the specific (orbital) quantum numbers have well-defined (unique) values. The alluded SRC states are described by the following wave functions taken in a φ — representation

$$\psi_m(\varphi) = (2\pi)^{-\frac{1}{2}}\, e^{im\varphi} \tag{5}$$

with the stipulations $\varphi \in [0, 2\pi)$ and $m = 0, \pm 1, \pm 2, \ldots$ The respective stipulations are required by the following facts. Firstly, in cases of SRC the angle φ is a ordinary polar coordinate which must satistfy the corresponding mathematical rules regarding the range of definition [28]. Secondly, from a physical perspective, in the same cases the wave function $\psi(\varphi)$ is enforced to have the property $\psi(0) = \psi(2\pi - 0) : = \lim_{\varphi \to 2\pi - 0} \psi(\varphi)$.

For the alluded SRC one finds

$$\Delta_\psi L_z = 0, \qquad \Delta_\psi \varphi = \frac{\pi}{\sqrt{3}} \,. \tag{6}$$

But these expressions for $\Delta_\psi L_z$ and $\Delta_\psi \varphi$ are incompatible with relation (4).

For avoiding the mentioned incompatibility many publications promoted the conception that in the case of $L_z - \varphi$ pair the RSUR (1) and the associated procedures of QM do not work correctly. Consequently it was accredited the idea that formula (4) must be proscribed and replaced by adjusted $\Delta_\psi L_z - \Delta_\psi \varphi$ relations planned to mime the RSUR (1). So, along the years, a lot of such mimic relations were proposed. In the main the respective relations can be expressed in one of the following forms:

$$\frac{\Delta_\psi L_z \cdot \Delta_\psi \varphi}{a\,(\Delta_\psi \varphi)} \geqslant \hbar \left| \langle b\,(\varphi) \rangle_\psi \right| , \tag{7}$$

$$\Delta_\psi L_z \cdot \Delta_\psi f(\varphi) \geqslant \hbar \left| \langle g(\varphi) \rangle_\psi \right| , \tag{8}$$

$$(\Delta_\psi L_z)^2 + \hbar^2 \left(\Delta_\psi u(\varphi)\right)^2 \geqslant \hbar^2 \left\langle v(\varphi) \right\rangle_\psi^2 , \tag{9}$$

$$\Delta_\psi L_z \cdot \Delta_\psi \varphi \geqslant \frac{\hbar}{2} \left| 1 - 2\pi \left| \psi(2\pi - 0) \right| \right| . \tag{10}$$

In (7)–(9) by a, b, f, g, u and v are denoted various adjusting functions (of $\Delta_\psi \varphi$ or of φ), introduced in literature by means of some circumstantial (and more or less fictitious) considerations.

Among the relations (7)–(10) of some popularity is (8) with $f(\varphi) = \sin\varphi$ (or $= \cos\varphi$) respectively $g(\varphi) = [\hat{L}_z, f(\hat{\varphi})]$. But, generally speaking, none of the respective relations is agreed unanimously as a suitable model able to substitute formula (4).

A minute examination of the facts shows that, in essence, the relations (7)–(10) are troubled by shortcomings revealed in the following remarks (***R***):

- ***R***$_1$: The relation (10) is correct from the usual perspective of QM (see formulas 18 and 25 in the next Secion). But the respective relation evidently does not mime the RSUR (1) presumed as standard within the mentioned restricted approaches of $L_z - \varphi$ problem;
- ***R***$_2$: Each replica from the classes depicted by (7)–(10) were planned to harmonize in a mimic fashion with the same presumed reference element represented by RSUR (1). But, in spite of such plannings, regarded comparatively, the respective replicas are not mutually equivalent;
- ***R***$_3$: Due to the absolutely circumstantial considerations by which they are introduced, the relations (7)–(9) are in fact ad hoc formulas without any direct descendence from general mathematics of QM. Consequently the respective relations ought to be appeciated by taking into account sentences such are:

  "*In … science, ad hoc often means the addition of corollary hypotheses or adjustment to a … scientific theory to save the theory from being falsified by compensating for anomalies not anticipated by the theory in its unmodified form. … Scientists are often suspicious or skeptical of theories that rely on … ad hoc adjustments*" [29].

  Then, if one wants to preserve the mathematical formalism of QM as a unitary theory, as it is accreditated in our days, the relations (7)–(9) must be regarded as unconvincing and inconvenient (or even prejudicial) elements;
- ***R***$_4$: In fact in relations (7)–(9) the angle φ is substituted more or less factitiously with the adjusting functions a, b, f, g, v or u. Then in fact , from a natural perspective of physics, such substitutions, and consequently the respective relations, are only mathematical artifacts. But, in physics, the mathematical artifacts burden the scientific discussions by additions of extraneous entities (concepts, assertions, reasonings, formulas) which are not associated with a true information regarding the real world. Then, for a good efficiency of the discussions, the alluded additions ought to be evaluated by taking into account the principle of parsimony: "*Entities should not be multiplied unneces-*

sarily" (known also [30, 31] as the "*Ockham's Razor*" slogan). Through such an evaluation the relations (7)–(9) appear as unnecessary exercises which do not give real and useful contributions for the elucidation of the $L_z-\varphi$ problem.

In our opinion the facts revealed in this Section offer a minimal but sufficient base for concluding that as regards the $L_z-\varphi$ problem the approaches restricted around the premises $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are unable to offer true and natural solutions.

3 The discordant examples with non-circular rotations

The discussions presented in the previous Section regard the situation of the $L_z-\varphi$ pair in relation with the mentioned SCR. But here is the place to note that the same pair must be considered also in connection with other orbital (spatial) motions which differ from SCR. Such motions are the non-circular rotations (NCR) . As examples of NCR we mention the quantum torsion pendulum (QTP) respectively the degenerate spatial rotations of the systems mentioned in the previous Section (i.e. a particle on a sphere, 2D or 3D rotators and an electron in a hydrogen atom). A rotation (motion) is degenerate if the energy of the system is well-specified while the non-energetic quantum numbers (here of orbital nature) take all permitted values.

From the class of NCR let us firstly refer to the case of a QTP which in fact is a simple quantum oscillator. Indeed a QTP which oscillates around the z-axis is characterized by the Hamiltonian

$$\hat{H} = \frac{1}{2I}\,\hat{L}_z^2 + \frac{1}{2}\,J\omega^2\varphi^2. \tag{11}$$

Note that in this expression φ denotes the azimuthal angle whose range of definition is the interval $(-\infty,\infty)$. In the same exppression appears $\hat{L}_z$ as the z-component of angular momentum operator defined also by (2). The other symbols J and ω in (11) represent the QTP momentum of inertia respectively the frequency of torsional oscillations. The Schrödinger equation associated to the Hamiltonian (11) shows that the QTP have eigenstates described by the wave functions

$$\psi_n(\varphi) = \psi_n(\xi) \propto \exp\left(-\frac{\xi^2}{2}\right)\mathcal{H}_n(\xi)\,,\quad \xi = \varphi\sqrt{\frac{J\omega}{\hbar}}\,, \tag{12}$$

where $n = 0, 1, 2, 3, \ldots$ signifies the oscillation quantum number and $\mathcal{H}_n(\xi)$ stand for Hermite polinomials of ξ. The eigenstates described by (12) have energies $E_n = \hbar\omega(n+\frac{1}{2})$. In the states (12) for the observables $L_z$and φ associated with the operators (2) one obtains the expressions

$$\Delta_\psi L_z = \sqrt{\hbar J\omega\left(n+\frac{1}{2}\right)}\,,\quad \Delta_\psi\varphi = \sqrt{\frac{\hbar}{J\omega}\left(n+\frac{1}{2}\right)}\,, \tag{13}$$

which are completely similar with the corresponding ones for the $x-p$ pair of a rectiliniar oscillator [27]. With the expressions (13) for $\Delta_\psi L_z$ and $\Delta_\psi\varphi$ one finds that in the case of QTP the $L_z-\varphi$ pair satisfies the proscribed formula (4).

From the same class of NCR let us now refer to a degenerate state of a particle on a sphere or of a 2D rotator. In such a state the energy is $E = \hbar^2\,l\,(l+1)/2J$ where the orbital number l has a well-defined value (J = moment of inertia). In the same state the magnetic number m can take all the values $-l, -l+1, \ldots, -1, 0, 1, \ldots, l-1, l$. Then the mentioned state is described by a wave function of the form

$$\psi_l(\theta,\varphi) = \sum_{m=-l}^{l} c_m\,Y_{lm}(\theta,\varphi)\,. \tag{14}$$

Here θ and φ denote polar respectively azimuthal angles ($\theta\in[0,\pi], \varphi\in[0,2\pi)$), $Y_{lm}(\theta,\varphi)$ are the spherical functions and c_m represent complex coefficients which satisfy the normalization condition $\sum_{m=-l}^{l}|c_m|^2 = 1$. With the expressions (2) for the operators $\hat{L}_z$ and $\hat{\varphi}$ in a state described by (14) one obtains

$$(\Delta_\psi L_z)^2 = \sum_{m=-l}^{l}|c_m|^2\,\hbar^2\,m^2 - \left[\sum_{m=-l}^{l}|c_m|^2\,\hbar m\right]^2, \tag{15}$$

$$(\Delta_\psi\,\varphi)^2 = \sum_{m=-l}^{l}\sum_{r=-l}^{l} c_m^*\,c_r\left(Y_{lm},\varphi^2\,Y_{lr}\right) - \left[\sum_{m=-l}^{l}\sum_{r=-l}^{l} c_m^*\,c_r\left(Y_{lm},\varphi Y_{lr}\right)\right]^2, \tag{16}$$

where (f,g) is the scalar product of the functions f and g.

By means of the expressions (15) and (16) one finds that in the case of alluded NCR described by the wave functions (14) it is possible for the proscribed formula (4) to be satisfied. Such a possibility is conditioned by the concrete values of the coefficients c_m.

Now is the place for the following remark

$\boldsymbol{R}_5$: As regards the $L_z-\varphi$ problem, due to the here revealed aspects, the NCR examples exceed the bounds of the presumptions $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ of usual restricted approaches. That is why the mentioned problem requires new approaches of general nature if it is possible.

4 A possible general appoach and some remarks associated with it.

A general approach of the $L_z-\varphi$ problem, able to avoid the shortcomings and discordances revealed in the previous two Sections, must be done by starting from the prime mathematical rules of QM. Such an approach is possible to be obtained as follows. Let us appeal to the usual concepts and notations of QM. We consider a quantum system whose state (of orbital nature) and two observables A_j ($j = 1, 2$) are described by the wave function ψ respectively by the operators $\hat{A}_j$. As usually with (f,g) we denote the scalar product of the functions

f and g . In relation with the mentioned state, the quantities $\langle A_j \rangle_\psi = (\psi\,, \hat{A}_j \psi)$ and $\delta_\psi \hat{A}_j = \hat{A}_j - \langle \hat{A}_j \rangle_\psi$ represent the mean (expected) value respectively the deviation-operator of the observable A_j regarded as a random variable. Then, by taking $A_1 = A$ and $A_2 = B$, for the two observables can be written the following Cauchy-Schwarz relation:

$$\left(\delta_\psi \hat{A}\psi,\, \delta_\psi \hat{A}\psi\right)\left(\delta_\psi \hat{B}\psi,\, \delta_\psi \hat{B}\psi\right) \geqslant \left|\left(\delta_\psi \hat{A}\psi,\, \delta_\psi B\psi\right)\right|^2. \quad (17)$$

For an observable A_j regarded as a random variable the quantity $\Delta_\psi A_j = \left(\delta_\psi \hat{A}_j \psi,\, \delta_\psi \hat{A}_j \psi\right)^{1/2}$ represents its standard deviation. From (17) it results directly that the standard deviations $\Delta_\psi A$ and $\Delta_\psi B$ of the observables A and B satisfy the relation

$$\Delta_\psi A \cdot \Delta_\psi B \geqslant \left|\left(\delta_\psi \hat{A}\psi,\, \delta_\psi B\psi\right)\right|, \quad (18)$$

which can be called *Cauchy-Schwarz formula* (CSF). Note that CSF (18) (as well as the relation (17) is always valid, i.e. for all observables, systems and states. Add here the important observation that the CSF (18) implies the restricted RSUR (1) only in the cases when the two operators $\hat{A} = \hat{A}_1$ and $\hat{B} = \hat{A}_2$ satisfy the conditions

$$\left(\hat{A}_j \psi, \hat{A}_k \psi\right) = \left(\psi, \hat{A}_j \hat{A}_k \psi\right), \quad j = 1, 2;\; k = 1, 2\,. \quad (19)$$

Indeed in such cases one can write the relation

$$\left(\delta_\psi \hat{A}\psi,\, \delta_\psi \hat{B}\psi\right) = \\ = \frac{1}{2}\left(\psi,\, \left(\delta_\psi \hat{A} \cdot \delta_\psi \hat{B}\psi + \delta_\psi \hat{B} \cdot \delta_\psi \hat{A}\right)\psi\right) - \\ - \frac{i}{2}\left(\psi,\, i\left[\hat{A},\, \hat{B}\right]\psi\right), \quad (20)$$

where the two terms from the right hand side are purely real and imaginary quantities respectively. Therefore in the mentioned cases from (18) one finds

$$\Delta_\psi A \cdot \Delta_\psi B \geqslant \frac{1}{2}\left|\left\langle [\hat{A},\, \hat{B}] \right\rangle_\psi\right|. \quad (21)$$

i.e. the well known RSUR (1). The above general framing of RSUR (1)/(21) shows that for the here investigated question of $L_z - \varphi$ pair it is important to examine the fulfilment of the conditions (19) in each of the considered cases. In this sense the following remarks are of direct interest.

R$_6$: In the cases described by the wave functions (5) for $L_z - \varphi$ pair one finds

$$\left(\hat{L}_z \psi_m,\, \hat{\varphi}\psi_m\right) = \left(\psi_m,\, \hat{L}_z \hat{\varphi}\psi_m\right) + i\hbar\,, \quad (22)$$

i.e. a clear violation in respect with the conditions (19);

R$_7$: In the cases associated with the wave functions (12) and (14) for $L_z - \varphi$ pair one obtains

$$\left(\hat{L}_z \psi_n,\, \hat{\varphi}\psi_n\right) = \left(\psi_n,\, \hat{L}_z \hat{\varphi}\psi_n\right), \quad (23)$$

$$\left(\hat{L}_z \psi_l,\, \hat{\varphi}\psi_l\right) = \left(\psi_l,\, \hat{L}_z \hat{\varphi}\psi_l\right) + \\ + i\hbar\left\{1 + 2\,\mathrm{Im}\left[\sum_{m=-l}^{l}\sum_{r=-l}^{l} c_m^* c_r \hbar m \left(Y_{lm},\, \hat{\varphi} Y_{lr}\right)\right]\right\}, \quad (24)$$

(where Im $[\alpha]$ denotes the imaginary part of α);

R$_8$: For any wave function $\psi(\varphi)$ with $\varphi \in [0, 2\pi)$ and $\psi(2\pi - 0) = \psi(0)$ it is generally true the formula

$$\left|\left(\delta_\psi \hat{L}_z\, \psi,\, \delta_\psi \hat{\varphi}\, \psi\right)\right| \geqslant \frac{\hbar}{2}\left|1 - 2\pi \left|\psi\left(2\pi - 0\right)\right|\right|, \quad (25)$$

which together with CSF (18) confirms relation (10).

The things mentioned above in this Section justify the following remarks

R$_9$: The CSF (18) is an ab origine element in respect with the RSUR (1)/(21). Moreover, (18) is always valid, independently if the conditions (19) are fulfilled or not;

R$_{10}$: The usual RSUR (1)/(21) are valid only in the circumstances strictly delimited by the conditions (19) and they are false in all other situations;

R$_{11}$: Due to the relations (22) in the cases described by the wave functions (5) the conditions (19) are not fulfilled. Consequently in such cases the restricted RSUR (1)/(21) are essentially inapplicable for the pairs $L_z - \varphi$. However one can see that in the respective cases, mathematically, the CSF (18) remains valid as a trivial equality $0 = 0$;

R$_{12}$: In the cases of NCR described by (12) the $L_z - \varphi$ pair satisfies the conditions (19) (mainly due to the relation (23). Therefore in the respective cases the RSUR (1)/(21) are valid for L_z and φ;

R$_{13}$: The fulfilment of the conditions (19) by the $L_z - \varphi$ pair for the NCR associated with (14) depends on the annulment of the second term in the right hand side from (24) (i.e. on the values of the coefficients c_m). Adequately, in such a case, the correctness of the corresponding RSUR (1)/(21) shows the same dependence;

R$_{14}$: The result (25) points out the fact that the adjusted relation (10) is only a secondary piece derivable fom the generally valid CSF (18);

R$_{15}$: The mimic relations (7)–(9) regard the cases with SCR described by the wave functions (5) when φ plays the role of polar coordinate. But for such a role [28] in order to be a unique (univocal) variable φ must be defined naturally only in the range $[0, 2\pi)$. The same range is considered in practice for the normalization of the wave functions (5). Therefore, in the cases under discussion the derivative with respect to φ refers to the mentioned range. Particularly for the extremities of the interval $[0, 2\pi)$ it has to operate with backward respectively forward derivatives. So in the alluded SCR cases the relations (2) and (3) act well, with a natural correctness. The same correctness is shown by the respective relations in connection with the NCR described by the wave functions (12) or (14). In fact, from a more general perspective, the relations (2) and (3) regard the QM operators $\hat{L}_z$ and $\hat{\varphi}$. Therefore they must have unique forms — i.e. expressions which do not depend on the particularities of the considered situations (e.g. systems with SCR or with NCR);

$\boldsymbol{R}_{16}$: The troubles of RSUR (1) regarding $L_z-\varphi$ pair are directly connected with the conditions (19). Then it is strange that in almost all the QM literature the respective conditions are not taken into account adequately. The reason seems to be related with the nowadays dominant Dirac's $<bra|$ and $|ket>$ notations. In the respective notations the terms from the both sides of (19) have a unique representation namely $<\psi|\hat{A}_j\,\hat{A}_k|\psi>$. The respective uniqueness can entail confusion (unjustified supposition) that the conditions (19) are always fulfiled. It is interesting to note that systematic investigations on the confusions/surprises generated by the Dirac's notations were started only recently [32]. Probably that further efforts on the line of such investigations will bring a new light on the conditions (19) as well as on other QM questions.

The ensemble of things presented above in this Section appoints a possible general approach for the discussed $L_z-\varphi$ problem and answer to a number of questions associated with the respective problem. Some significant aspects of the respective approach are noted in the next Section.

5 Conclusions

The facts and arguments discussed in the previous Sections guide to the following conclusions ($\boldsymbol{C}$):

$\boldsymbol{C}_1$: For the $L_z-\varphi$ pair the relations (2)–(3) are always viable in respect with the general CSF (18). That is why, from the QM perspective, for a correct description of questions regarding the respective pair, it is not at all necessary to resort to the mimetic formulas (7)–(10). Eventually the respective formulas can be accounted as ingenious execises of pure mathematical facture. An adequate description of the mentioned kind can be given by taking CSF (18) and associated QM procedures as basic elements;

$\boldsymbol{C}_2$: In respect with the conjugated observables L_z and φ the RSUR (1)/(21) is not adequate for the role of reference element for normality . For such a role the CSF (18) is the most suitable. In some cases of interest the respective CSF degenerates in the trivial equality $0=0$;

$\boldsymbol{C}_3$: In reality the usual procedures of QM, illustrated above by the relations (2), (3), (17) and (18), work well and without anomalies in all situations regarding the $L_z-\varphi$ pair. Consequently with regard to the conceptual as well as practical interests of science the mimic relations like (7)–(9) appear as useless inventions.

Now we wish to add the following observations ($\boldsymbol{O}$):

$\boldsymbol{O}_1$: Mathematically the relation (17) is generalisable in the form

$$\det\left[\left(\delta_\psi\hat{A}_j\psi,\,\delta_\psi\hat{A}_k\psi\right)\right]\geqslant 0 \qquad (26)$$

where $\det[\alpha_{jk}]$ denotes the determinant with elements α_{jk} and $j=1,2,\ldots,r$; $k=1,2,\ldots,r$ with $r\geqslant 2$. Such a form results from the fact that the quantities $\left(\delta_\psi\hat{A}_j\psi,\,\delta_\psi\hat{A}_k\psi\right)$ constitute the elements of a Hermitian and non-negatively defined matrix. Newertheless, comparatively with (17), the generalisation (26) does not bring supplementary and inedited features regarding the conformability of observables $L_z-\varphi$ with the mathematical rules of QM;

$\boldsymbol{O}_2$: We consider [34, 42] that the above considerations about the problem of $L_z-\varphi$ pair can be of some nontrivial interest for a possible revised approach of the similar problem of the pair N — ϕ (number-phase) which is also a subject of controversies in recent publications (see [4, 11, 12, 13, 35, 36, 37, 38, 39] and References therein);

$\boldsymbol{O}_3$: Note that we have limited this paper only to mathematical aspects associated with the RSUR (1), without incursions in debates about the interpretations of the respective RSUR. Some opinions about those interpretations and connected questions are given in [40, 41, 42]. But the subject is delicate and probably that it will rouse further debates.

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