Detection of the Relativistic Corrections to the Gravitational Potential using a Sagnac Interferometer

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General Relativity predicts the existence of relativistic corrections to the static Newtonian potential which can be calculated and verified experimentally. The idea leading to quantum corrections at large distances is that of the interactions of massless particles which only involve their coupling energies at low energies. In this short paper we attempt to propose the Sagnac intrerferometric technique as a way of detecting the relativistic correction suggested for the Newtonian potential, and thus obtaining an estimate for phase difference using a satellite orbiting at an altitude of 250 km above the surface of the Earth.

1 Introduction

The potential acting between to masses M and m that separated from their centers by a distance r is:

$$V(r) = -\frac{GMm}{r}\,, \tag{1}$$

where s the Newton's constant of gravitation. This potential is of course only approximately valid [1]. For large masses and or large velocities the theory of General Relativity predicts that there exist relativistic corrections which can be calculated and also verified experimentally [2]. In the microscopic distance domain, we could expect that quantum mechanics, would predict a modification in the gravitational potential in the same way that the radiative corrections of quantum electrodynamics leads to a similar modification of the Coulombic interaction [3].

Even though the theory of General Relativity constitutes a very well defined classical theory, it is still not possible to combine it with quantum mechanics in order to create a satisfied theory of quantum gravity. One of the basic obstacles that prevent this from happening is that General Relativity does not actually fit the present paradigm for a fundamental theory that of a renormalizable quantum field theory. Gravitational fields can be successfully quantized on smooth-enough space-times [4], but the form of gravitational interactions is such that they induce unwanted divergences which can not be absorbed by the renormalization of the parameters of the minimal General Relativity [5]. Somebody can introduce new coupling constants and absorb the divergences then, one is unfortunately led to an infinite number of free parameters. In spite the difficulty above quantum gravity calculations can predict long distance quantum corrections.

The main idea leading to quantum corrections at large distances is due to the interactions of massless particles which only involve their coupling energies at low energies, something that it is known from the theory of General Relativity, even though at short distances the theory of quantum gravity differs resulting to finite correction of the order, $O\left(\frac{G\hbar}{c^3 r^3}\right)$. The existence of a universal long distance quantum correction to the Newtonian potential should be relevant for a wide class of gravity theories. It is well known that the ultraviolet behaviour of Einstein's pure gravity can be improved, if higher derivative contributions to the action are added, which in four dimensions take the form:

$$\alpha R^{\kappa\lambda} R_{\kappa\lambda} + \beta R^2, \tag{2}$$

where α and β are dimensionless coupling constants. What makes the difference is that the resulting classical and quantum corrections to gravity are expected to significantly alter the gravitational potential at short distances comparable to that of Planck length $\ell_P = \sqrt{\frac{G\hbar}{c^3}} = 10^{-35}$ m, but it should not really affect its behaviour at long distances. At long distances is the structure of the Einstein-Hilbert action that actually determines that. At this point we should mentioned that some of the calculation to the corrections of the Newtonian gravitational potential result in the absence of a cosmological constant Λ which usually complicates the perturbative treatment to a significant degree due to the need to expand about a non-flat background.

In one loop amplitude computation one needs to calculate all first order corrections in G, which will include both the relativistic $O\left(\frac{G^2 m^2}{c^2}\right)$ and the quantum mechanical $O\left(\frac{G\hbar}{c^3}\right)$ corrections to the classical Newtonian potential [6].

2 The corrections to the potential

Our goal is not to present the details of the one loop treatment that leads to the corrections of the Newtonian gravitational potential but rather sate the result and then use it in our calculations. Valid in order of G^2 we have that the corrected potential now becomes [6]:

$$V(r) = -\frac{GMm}{r}\left[1 - \frac{G(M+m)}{2c^2 r} - \frac{122\,G\hbar}{15\pi c^3 r^2}\right]. \tag{3}$$

Observing (3) we see that in the correction of the static Newtonian potential two different length scales are involved. First, the Planck length $\ell_P = \sqrt{\frac{G\hbar}{c^3}} = 10^{-35}$ m and second the Schwarzschild radii of the heavy sources $r_{sch} = \frac{2GM_n}{c^2}$. Furthermore there are two independent dimensionless parameters which appear in the correction term, and involve the ratio of these two scales wit respect to the distance r. Presumably for meaningful results the two length scales are much smaller than r.

3 Perturbations due to oblateness J_2

Because the Earth's gravitational potential is not that of a perfect spherical body, we can approximate its potential as a spherical harmonic expansion of the following form:

$$V(r,\phi) = -\frac{GMm}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R_e}{r}\right)^n P_n(\sin\phi)\right] = \\ = \frac{GMm}{r}\left[V_o + V_{J_2} + V_{J_3} + \ldots\right], \tag{4}$$

where:

r = geocentric distance,

ϕ = geocentric latitude.

R_e = means equatorial radius of the Earth,

P_n = Legendre polynomial of degree n and order zero,

$J_n = J_{n_0}$ jonal harmonics of order zero, that depend on the latitude ϕ only,

and the first term GMm/r now describes the potential of a homogeneous sphere and thus refers to Keplerian motion, the remaining part represents the Earth's oblateness via the zonal harmonic coefficients and [7]

$$\left.\begin{aligned} V_0 &= -1 \\ V_{J_2} &= \frac{J_2}{2}\left(\frac{R_e}{r}\right)^2\left(3\sin^2\phi - 1\right) \\ V_{J_3} &= \frac{J_3}{2}\left(\frac{R_e}{r}\right)^3\left(5\sin^3\phi - 3\sin\phi\right) \end{aligned}\right\} \tag{5}$$

similarly [8]

$$\left.\begin{aligned} J_2 &= 1{,}082.6\times10^{-6} \\ J_3 &= -2.53\times10^{-6} \end{aligned}\right\}. \tag{6}$$

Therefore equation (4) can be further written:

$$V(r,\phi) = -\frac{GMm}{r}\times \\ \times\left[1 - \sum_{n=2}^{\infty} J_n\left(\frac{R_e}{r}\right)^n P_n(\sin\phi) - \frac{G(M+m)}{2rc^2}\right] = \\ = \frac{GMm}{r}\left[V_o + V_{J_2} + V_{J_3} + \ldots\; - V_{Relativistic}\right]. \tag{6a}$$

Since J_2 is 400 larger that any other J_n coefficients, we can disregard them and write the following expression for the Earth's potential function including only the relativistic correction and omitting the quantum corrections as being very small we have:

$$V(r,\phi) = -\frac{GM_e m}{r} + \frac{GM_e m R_e^2 J_2}{r^3}\left(\frac{3}{2}\sin^2\phi - \frac{1}{2}\right) + \\ + \frac{G^2 M_e m(M_e+m)}{c^2 r^2}. \tag{7}$$

Since we propose a satellite in orbit that carries the Sagnac instrument it will of a help to express equation (7) for the potential in terms of the orbital elements. We know that $\sin\phi = \sin i\,\sin(f+\omega)$ where i is the inclination of the orbit, f is the true anomaly and ω is the argument of the perigee. Ignoring long and short periodic terms (those containing ω and f) we write (7) in terms of the inclination as follows:

$$V(r,\phi) = -\frac{GM_e m}{r} + \frac{3GM_e m R_e^2 J_2}{2r^3}\left(\frac{\sin^2 i}{2} - \frac{1}{3}\right) + \\ + \frac{G^2 M_e m(M_e+m)}{c^2 r^2} \tag{8}$$

therefore the corresponding total acceleration that a mass m at $r > R_e$ has becomes:

$$g_{tot} = -\frac{1}{m}\frac{\partial}{\partial r}\left[-\frac{GM_e m}{r} + \frac{3GM_e m R_e^2 J_2}{2r^3}\times \\ \times\left(\frac{\sin^2 i}{2} - \frac{1}{3}\right) + \frac{G^2 M_e m(M+m)}{r^2 c^2}\right] \tag{9}$$

so that:

$$g_{tot} = -\frac{GM_e}{r^2} + \frac{9GM_e R_e^2 J_2}{2r^4}\left(\frac{\sin^2 i}{2} - \frac{1}{3}\right) + \\ + \frac{G^2 M_e(M_e+m)}{c^2 r^3}. \tag{10}$$

4 Basic Sagnac interferometric theory

The Sagnac interferometer is based on the *Sagnac effect*, reported by G. Sagnac in 1913 [8]. Two beams are sent in opposite directions around the interferometer until they meet

$$\Delta\phi_{rs} = \frac{8\pi^2 R_s^2 N\Omega_s \nu \left(R_s\Omega_s + \pi^{-2} v_{orb}\right)}{\left(c^2 - R_s^2\Omega_s^2\right)\left[1 - \frac{R_s}{c^2}\left(-\frac{GM_e}{r^2} + \frac{9GM_e R_e^2 J_2}{2r^4}\left(\frac{\sin^2 i}{2} - \frac{1}{3}\right) + \frac{G^2M_e^2}{r^3c^2}\right)\left[1 - \cos\left[\frac{2\pi R_s\Omega_s}{c}\left(1 + \frac{R_s\Omega_s}{c}\right)^{-1}\right]\right]\right]} \tag{15}$$

$$\Delta\phi_{rs} = \frac{8\pi^2 R_s^2 N\Omega_s \nu \left(R_s\Omega_s + \pi^{-2}\sqrt{\frac{GM_e}{(R_e+z_{orb})}}\right)}{\left(c^2 - R_s^2\Omega_s^2\right)\left[1 + \frac{R_s}{c^2}\left(\frac{GM_e}{r^2} - \frac{3GM_e R_e^2 J_2}{4r^4} - \frac{G^2M_e^2}{r^3c^2}\right)\left[1 - \cos\left[\frac{2\pi R_s\Omega_s}{c}\left(1 + \frac{R_s\Omega_s}{c}\right)^{-1}\right]\right]\right]} \tag{16}$$

$$\Delta\phi_{rs} = \frac{8\pi^2 R_s^2 N\Omega_s \nu \left(1 + \frac{R_s^2\Omega_s^2}{c^2}\right)\left(R_s\Omega_s + \pi^{-2}\sqrt{\frac{GM_e}{R_e}\left(1 - \frac{z_{orb}}{R_e}\right)}\right)}{c^2\left[1 + R_s\left(\frac{GM_e}{c^2R_e^2}\left(1 - \frac{2z_{orb}}{R_e}\right) - \frac{3GM_e R_e^2 J_2}{4c^2R_e^4}\left(1 - \frac{4z_{orb}}{R_e}\right) - \frac{G^2M_e^2}{R_e^3c^4}\left(1 - \frac{3z_{orb}}{R_e}\right)\right)\left[1 - \cos\left[\frac{2\pi R_s\Omega_s}{c}\left(1 + \frac{R_s\Omega_s}{c}\right)^{-1}\right]\right]\right]} \tag{17}$$

again to create a phase pattern. By rotating the interferometer in the direction of either the clockwise (CW) or counter-clockwise (CCW) beam, a phase difference results between the two beams that its given by:

$$\Delta\Phi_{rs} = \frac{8\pi^2 R_{sag}^2 N\Omega\nu}{\left(c^2 - a^2\Omega^2\right)}\,, \tag{11}$$

where Ω is the angular velocity of the interferometer, R_{sag} is the radius of the interferometer, N is the number of turns of fiber around the radius and ν is the frequency of light in the fiber.

Let us now assume that the Sagnac interferometer and its light laser beams are in the region of space around the Earth where the gravitational potential is given by equation (3) and let us further assume that the quantum correction to the potential is really negligible. If the Sagnac light loop area has a unit vector that is perpendicular to the acceleration of gravity vector, then the motion of the interferometer will exhibit a red-shift that will be given by:

$$f_{rs} = \frac{f}{1 - \frac{\Delta V}{c^2}} = \frac{f}{1 - \frac{g_{cor} z}{c^2}}\,, \tag{12}$$

where ΔV is the difference in the potential between to different points P_1 and P_2, and g_{cor} is the corrected or total acceleration of gravity and z is the difference in vertical distance between the two beams as the interferometer coil rotates. This distance z that the laser beams see is given by:

$$z = R_{sag}\left\{1 - \cos\left[\frac{2\pi\Omega R_{sag}}{c\left(1 + \frac{R_{sag}\Omega}{c}\right)}\right]\right\}. \tag{13}$$

This Sagnac effect can also be amplified by an interferometer that is in orbit, where the orbital velocity of the interferometer with respect to the Earth's surface produces an increased phase shift. Both terms involved in the acceleration of gravity in the first one:

$$\Delta\Phi_{rs} = \frac{8\pi^2 R_{sag}^2 N\Omega\nu\left(R_{sag}\Omega + \frac{v_{orb}}{\pi^2}\right)}{\left(c^2 - R_{sag}^2\Omega^2\right)\left[1 - \frac{g_{tot} z}{c^2}\right]} \tag{14}$$

using (14) and taking into account that $M \gg m$ we further obtain (15), where M is the source of the gravitational field = the mass of the Earth in our case M_e, and R is the radius of the massive body $= R_e$, and $r = R_e + z_{orb}$ it's orbital height plus Earth radius for an Earth-based satellite.

This Sagnac effect can also be amplified by an interferometer that is in orbit, where the orbital velocity of the interferometer with respect to the Earth's surface produces an increased phase shift. Both terms involved in the acceleration of gravity in the first one:

5 Sagnac in circular orbit of known inclination

Let now a Sagnac interferometer be aboard a satellite in a circular polar orbit of inclination $i = 90$ degrees. If the inclination is 90 degrees the term $\sin^2\frac{i}{2} - \frac{1}{3} = \frac{1}{6}$ and the orbital velocity at some height z above the surface of the Earth is $v_{orb} = \sqrt{\frac{GM_e}{(R_e+Z_{orb})}}$ and (6) takes the form (16) can be finally written as (17).

6 Sagnac in elliptical orbit of known inclination

If now a satellite is carrying a Sagnac device is in an elliptical orbit of eccentricity e and semi-major axis a we have that the radial orbital vector and the orbital velocity are given by:

$$r(f) = \frac{a\left(1 - e^2\right)}{1 + e\cos f}\,, \tag{18}$$

$$v^2 = GM_e\left(\frac{2}{r} - \frac{1}{a}\right) = \frac{GM_e}{a}\left[\frac{2\left(1 + e\cos f\right)}{\left(1 - e^2\right)} - 1\right], \tag{19}$$

where f is the true anomaly of the orbit. Substituting now in (8) we obtain (20).

If we use the fact that $GM_e = n^2a^3$ where n is the mean motion of the satellite, equation (20) can be further written as (21).

When the satellite approaches perigee its orbital velocity will increase, so we will expect to see a higher phase difference than any other point of the orbit, and similarly the effect

$$\Delta\phi_{rs} = \frac{8\pi^2 R_s^2 N\Omega_s \nu \left(1+\frac{R_s^2\Omega_s^2}{c^2}\right)\left(R_s\Omega_s + \pi^{-2}\sqrt{\frac{GM_e}{a}\left(\frac{1+e^2+2e\cos f}{1-e^2}\right)}\right)}{c^2\left[1+R_s\left(\frac{GM_e(1+e\cos f)^2}{c^2a^2(1-e^2)^2} - \frac{3GM_eR_e^2J_2(1+e\cos f)^4}{4c^2a^4(1-e^2)^4} - \frac{G^2M_e^2(1+e\cos f)^3}{c^4a^3(1-e^2)^3}\right)\left[1-\cos\left[\frac{2\pi R_s\Omega_s}{c}\left(1+\frac{R_s\Omega_s}{c}\right)^{-1}\right]\right]\right]} \tag{20}$$

$$\Delta\phi_{rs} = \frac{8\pi^2 R_s^2 N\Omega_s \nu \left(1+\frac{R_s^2\Omega_s^2}{c^2}\right)\left(R_s\Omega_s + \pi^{-2}na\sqrt{\left(\frac{1+e^2+2e\cos f}{1-e^2}\right)}\right)}{c^2\left[1+R_s\left(\frac{n^2a(1+e\cos f)^2}{c^2(1-e^2)^2} - \frac{3n^2R_e^2J_2(1+e\cos f)^4}{4c^2a(1-e^2)^4} - \frac{n^4a^3(1+e\cos f)^3}{c^4(1-e^2)^3}\right)\left[1-\cos\left[\frac{2\pi R_s\Omega_s}{c}\left(1+\frac{R_s\Omega_s}{c}\right)^{-1}\right]\right]\right]} \tag{21}$$

$$\Delta\phi_{rs}\,(perigee) = \frac{8\pi^2 R_s^2 N\Omega_s \nu \left(1+\frac{R_s^2\Omega_s^2}{c^2}\right)\left(R_s\Omega_s + \pi^{-2}\sqrt{\frac{GM_e}{a}\left(\frac{1+e}{1-e}\right)}\right)}{c^2\left[1+R_s\left(\frac{GM_e}{c^2a^2(1-e)^2} - \frac{3GM_eR_e^2J_2}{4c^2a^4(1-e)^4} - \frac{G^2M_e^2}{c^4a^3(1-e)^3}\right)\left[1-\cos\left[\frac{2\pi R_s\Omega_s}{c}\left(1+\frac{R_s\Omega_s}{c}\right)^{-1}\right]\right]\right]} \tag{24}$$

$$\Delta\phi_{rs}\,(perigee) = \frac{8\pi^2 R_s^2 N\Omega_s \nu \left(1+\frac{R_s^2\Omega_s^2}{c^2}\right)\left(R_s\Omega_s + \pi^{-2}na\sqrt{\frac{1+e}{1-e}}\right)}{c^2\left[1+R_s\left(\frac{n^2a}{c^2(1-e)^2} - \frac{3n^2R_e^2J_2}{4c^2a(1-e)^4} - \frac{n^4a^3}{c^4(1-e)^3}\right)\left[1-\cos\left[\frac{2\pi R_s\Omega_s}{c}\left(1+\frac{R_s\Omega_s}{c}\right)^{-1}\right]\right]\right]} \tag{25}$$

$$\Delta\phi_{rs}\,(apogee) = \frac{8\pi^2 R_s^2 N\Omega_s \nu \left(1+\frac{R_s^2\Omega_s^2}{c^2}\right)\left(R_s\Omega_s + \pi^{-2}\sqrt{\frac{GM_e}{a}\left(\frac{1-e}{1+e}\right)}\right)}{c^2\left[1+R_s\left(\frac{GM_e}{c^2a^2(1+e)^2} - \frac{3GM_eR_e^2J_2}{4c^2a^4(1+e)^4} - \frac{G^2M_e^2}{c^4a^3(1+e)^3}\right)\left[1-\cos\left[\frac{2\pi R_s\Omega_s}{c}\left(1+\frac{R_s\Omega_s}{c}\right)^{-1}\right]\right]\right]} \tag{26}$$

$$\Delta\phi_{rs}\,(apogee) = \frac{8\pi^2 R_s^2 N\Omega_s \nu \left(1+\frac{R_s^2\Omega_s^2}{c^2}\right)\left(R_s\Omega_s + \pi^{-2}na\sqrt{\frac{1+e}{1-e}}\right)}{c^2\left[1+R_s\left(\frac{n^2a}{c^2(1+e)^2} - \frac{3n^2R_e^2J_2}{4c^2a(1+e)^4} - \frac{n^4a^3}{c^4(1+e)^3}\right)\left[1-\cos\left[\frac{2\pi R_s\Omega_s}{c}\left(1+\frac{R_s\Omega_s}{c}\right)^{-1}\right]\right]\right]} \tag{27}$$

will be minimum at the point of apogee because the satellite's velocity is minimal. The distance at perigee and apogee are given by the equations below:

$$\left.\begin{aligned} r_{pg} &= a\,(1-e) \\ r_{apg} &= a\,(1+e) \end{aligned}\right\} \tag{22}$$

also the corresponding velocities are:

$$\left.\begin{aligned} v_{pg}^2 &= \frac{GM}{a}\left(\frac{1+e}{1-e}\right) \\ v_{apg}^2 &= \frac{GM_e}{a}\left(\frac{1-e}{1+e}\right) \end{aligned}\right\}, \tag{23}$$

therefore the phase difference detected by the Sagnac due to the contribution of the Earth's oblateness plus relativistic correction to the potential at perigee and apogee can be written as (24) or again (25).

Similarly the phase difference at apogee can be written as (26) or again (27).

For this last case of the elliptical orbit in (25) and (26) where the Sagnac interferometer is on the satellite and we assume $R_s = 1$ m, $\nu = 2\times10^{14}$ Hz, $N = 10^6$, $\Omega_s = 400$ rad/sec, $a = 8\times10^6$ m, $e = 0.2$, $R_e = 6.378\times10^6$ meters we arrive at the following values for $\Delta\phi$:

$\Delta\phi\,(perigee) = 3.57\times10^{-16}$ radians,

$\Delta\phi\,(apogee) = 2.44\times10^{-16}$ radians.

These values are based on the dominant potential correction in (11) of section 3 which is the first term in (11) or the Newtonian correction:

Newtonian correction $= 2.17\times10^{-16}$ radians.

In comparison, the second and third terms in (11) are the oblateness and relativistic corrections respectively and they produce the following values based on the given parameters:

Oblateness correction $= 8.52\times10^{-20}$,

Relativistic correction $= 7.91\times10^{-26}$.

So by comparison of the values above, the Newtonian correction is much easier to measure.

$$\Delta\phi_{rs} = \frac{8\pi^2 a_s^2 N \Omega_s \nu \left(a_s \Omega_s + \pi^{-2} \sqrt{\frac{GM_e}{(R_e + z_{orb})}} \right)}{\left(c^2 - a_s^2 \Omega_s^2\right) \left[1 + \frac{a_s \left[1 + (e^2 + e - 1) \cos\left[\frac{2\pi a_s \Omega_s}{c} \left(1 + \frac{a_s \Omega_s}{c}\right)^{-1} \right] \right]}{\left(1 + e \cos\left[\frac{2\pi a_s \Omega_s}{c} \left(1 + \frac{a_s \Omega_s}{c}\right)^{-1} \right]\right)} \left(\frac{GM_e}{r^2 c^2} - \frac{3GM_e R_e^2 J_2}{4 r^4 c^2} - \frac{G^2 M_e^2}{r^3 c^4} \right) \right]} \tag{30}$$

$$\Delta\phi_{rs} = \frac{8\pi^2 a_s^2 N \Omega_s \nu \left(1 + \frac{a_s^2 \Omega^2}{c^2}\right) \left(a_s \Omega_s + \pi^{-2} \sqrt{\frac{GM_e}{R_e} \left(1 - \frac{z_{orb}}{R_e}\right)} \right)}{c^2 \left[1 + \frac{a_s \left[1 + (e^2 + e - 1) \cos\left[\frac{2\pi a_s \Omega_s}{c} \left(1 + \frac{a_s \Omega_s}{c}\right)^{-1} \right] \right]}{\left(1 + e \cos\left[\frac{2\pi a_s \Omega_s}{c} \left(1 + \frac{a_s \Omega_s}{c}\right)^{-1} \right]\right)} \left(\frac{GM_e}{R_e^2 c^2} \left(1 - \frac{2 z_{orb}}{R_e}\right) - \frac{3GM_e R_e^2 J_2}{4 R_e^4 c^2} \left(1 - \frac{4 z_{orb}}{R_e}\right) - \frac{G^2 M_e^2}{R_e^3 c^4} \left(1 - \frac{3 z_{orb}}{R_e}\right) \right) \right]} \tag{31}$$

$$\Delta\phi_{rs} = \frac{8\pi^2 a_s^2 N \Omega_s \nu \left(1 + \frac{a_s^2 \Omega_s^2}{c^2}\right) \left(a_s \Omega_s + \pi^{-2} \sqrt{\frac{GM_e}{a} \left(\frac{1 + e^2 + 2e \cos f}{1 - e^2} \right)} \right)}{c^2 \left[1 + \frac{a_s \left[1 + (e^2 + e - 1) \cos\left[\frac{2\pi a_s \Omega_s}{c} \left(1 + \frac{a_s \Omega_s}{c}\right)^{-1} \right] \right]}{\left(1 + e \cos\left[\frac{2\pi a_s \Omega_s}{c} \left(1 + \frac{a_s \Omega_s}{c}\right)^{-1} \right]\right)} \left(\frac{GM_e (1 + e\cos f)^2}{c^2 a^2 (1 - e^2)^2} - \frac{3GM_e R_e^2 J_2 (1 + e \cos f)^4}{4 c^2 a^4 (1 - e^2)^4} - \frac{G^2 M_e^2 (1 + e\cos f)^3}{c^4 a^3 (1 - e^2)^3} \right) \right]} \tag{32}$$

$$\Delta\phi_{rs}\,(perigee) = \frac{8\pi^2 a_s^2 N \Omega_s \nu \left(1 + \frac{a_s^2 \Omega_s^2}{c^2}\right) \left(a_s \Omega_s + \pi^{-2} \sqrt{\frac{GM_e}{a} \left(\frac{1 + e}{1 - e} \right)} \right)}{c^2 \left[1 + \frac{a_a \left[1 + (e^2 + e - 1) \cos\left[\frac{2\pi a_s \Omega s}{c} \left(1 + \frac{a_s \Omega_s}{c}\right)^{-1} \right] \right]}{\left(1 + e \cos\left[\frac{2\pi a_s \Omega_s}{c} \left(1 + \frac{a_s \Omega_s}{c}\right)^{-1} \right]\right)} \left(\frac{GM_e}{c^2 a^2 (1 - e)^2} - \frac{3GM_e R_e^2 J_2}{4 c^2 a^4 (1 - e)^4} - \frac{G^2 M_e^2}{c^4 a^3 (1 - e)^3} \right) \right]} \tag{33}$$

$$\Delta\phi_{rs}\,(apogee) = \frac{8\pi^2 a_s^2 N \Omega_s \nu \left(1 + \frac{a_s^2 \Omega_s^2}{c^2}\right) \left(a_s \Omega_s + \pi^{-2} \sqrt{\frac{GM_e}{a} \left(\frac{1 - e}{1 + e} \right)} \right)}{c^2 \left[1 + \frac{a_s \left[1 + (e^2 + e - 1) \cos\left[\frac{2\pi a_s \Omega_s}{c} \left(1 + \frac{a_s \Omega_s}{c}\right)^{-1} \right] \right]}{\left(1 + e \cos\left[\frac{2\pi a_s \Omega_s}{c} \left(1 + \frac{a_s \Omega_s}{c}\right)^{-1} \right]\right)} \left(\frac{GM_e}{c^2 a^2 (1 + e)^2} - \frac{3GM_e R_e^2 J_2}{4 c^2 a^4 (1 + e)^4} - \frac{G^2 M_e^2}{c^4 a^3 (1 + e)^3} \right) \right]} \tag{34}$$

The $\Delta\phi$ values given above may be more easily measured using a QPSK-modulator inserted in the CCW or CW beam path to improve phase resolution. Also, the use of higher wavelengths (factor of 10 higher in frequency) will increase resolution.

7 We suggest a Sagnac with an elliptic fiber loop

To attempt increasing the resolution of the phase difference of the Sagnac interferometer let us now propose a Sagnac loop, that has the shape of an ellipse that rotates with an angular velocity Ω. In this case it can be shown that the height difference between two points on the ellipse can be given by:

$$z = a \left[\frac{1 + \left(e^2 + e - 1\right) \cos\theta}{1 + e\cos\theta} \right]. \tag{28}$$

To check the validity of the formula we derived we can set e=0 which is the case of a circular Sagnac fiber optical path we can see that the (13) in now retrieved since

$R_{sag} = a_{loop(sag)} = a_s$ is the semi major axis of the elliptical fiber loop. When the ellipse spins with angular velocity Ω that would force it to trace out a circle whose radius r, will be that of the semi-major axis a of the ellipse, and therefore we can finally write for (13):

$$z = \frac{a_s \left[1 + (e^2 + e - 1) \cos\left\{ \frac{2\pi a_s \Omega_s}{c} \left(1 + \frac{a_s \Omega_s}{c}\right)^{-1} \right\} \right]}{\left(1 + e\cos\left\{ \frac{2\pi a_s \Omega_s}{c} \left(1 + \frac{a_s \Omega_s}{c}\right)^{-1} \right\} \right)}. \tag{29}$$

8 Circular orbit formula for the phase difference of the Sagnac

Let now as before have a Sagnac interferometer be aboard a satellite in a circular polar orbit of inclination $i = 90$ degrees. If the inclination is 90 degrees the term $\sin^2\frac{i}{2} - \frac{1}{3} = \frac{1}{6}$ and the orbital velocity at some height z above the surface of the Earth is $v_{orb(circ)} = \sqrt{\frac{GM_e}{(R_e + z_{orb})}}$ and (6) takes the form (30) that can be finally written as (31).

9 Sagnac in elliptical orbit of known inclination

If now a satellite is carrying a Sagnac device is in an elliptical orbit of eccentricity e and semi-major axis a we have that the radial orbital vector and the orbital velocity are given by (32).

At perigee the equation (32) becomes (33) and also (34).

For (33) and (34) above the following values are computed assuming $e = 0.2$, $\nu = 2\times10^{14}$ Hz, $a = 8\times10^{6}$ meters, $N = 1$ (because the orbit is the Sagnac loop), $R_{sag} = R_{perigee}$ or R_{apogee} as determined by (22), $\Omega_{perigee} = 0.001$ rad/sec, and $\Omega_{apogee} = 6\times10^{-4}$ rad/sec we find,

$$\Delta\phi\,(perigee) = 6.05\times10^{10} \text{ radians},$$

$$\Delta\phi\,(apogee) = 2.36\times10^{10} \text{ radians}.$$

These values are for measuring the dominant Newtonian contribution as described in Section 6. To detect relativistic contribution which is 3.64×10^{-10} smaller than the Newtonian contribution the corresponding phase-shifts from (33) and (34) are:

$$\Delta\phi\,(perigee) = 22 \text{ radians},$$

$$\Delta\phi\,(apogee) = 8.59 \text{ radians}.$$

Thus, the relativistic contribution in (11) of Section 3 is easily measurable using a Sagnac interferometer where the satellites in orbit are the Sagnac loop. In this scenario, the light path can be implemented by transmitting laser beams from one satellite to the next satellite in orbit ahead of it. Also, by using the maximum spacing possible between satellites in orbit this will allow line of site transmission while reducing the number of satellites required for the Sagnac loop. With the potential to measure such small relativistic corrections, the merit of using satellites to implement a large Sagnac loop of radius $R_s = R_{ap}$ or R_{per} is well worth considering.

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