# An Asymptotic Solution for the Navier-Stokes Equation

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We have used as the velocity field of a fluid the functional form derived in Casuso (2007), obtained by studying the origin of turbulence as a consequence of a new description of the density distribution of matter as a modified discontinuous Dirichlet integral. As an interesting result we have found that this functional form for velocities is a solution to the Navier-Stokes equation when considering asymptotic behaviour, i.e. for large values of time.

## 1 Introduction

The Euler and Navier-Stokes equations describe the motion of a fluid. These equations are to be solved for an unknown velocity vector $\vec{u}(\vec{r},t)$ and pressure $P(\vec{r},t)$, defined for position $\vec{r}$ and time $t \geqslant 0$. We restrict attention here to incomprenssible fluids filling all real space. Then the Navier-Stokes equations are: a) Newton's law $\vec{f}=m\vec{a}$ for a fluid element subject to the external force $\vec{g}$ (gravity) and to the forces arising from pressure and friction, and b) The condition of incompressibility. A fundamental problem in the analysis is to find any physically reasonable solution for the Navier-Stokes equation, and indeed to show that such a solution exists. Many numerical computations appear to exhibit blowup for solutions of the Euler equations (the same as Navier-Stokes equations but for zero viscosity), but the extreme numerical instability of the equations makes it very hard to draw reliable conclusions (see Bertozzi and Majda 2002 [1]). Important progress has been made in understanding weak solutions of the Navier-Stokes equations (Leray 1934 [2], Khon and Nirenberg 1982 [3], Scheffer 1993 [4], Schnirelman 1997 [5], Caffarelli and Lin 1998 [6]). This type of solutions means that one integrates the equation against a test function, and then integrates by parts to make the derivatives fall on the test function. In the present paper we test directly the validity of a solution which was obtained previously from the study of turbulence.

## 2 Demonstration of validity of the asymptotic solution

We start from the Navier-Stokes equation for one-dimension:

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} = \nu \frac{\partial^2 u_x}{\partial x^2} - \frac{\partial P}{\partial x} + g\,, \tag{1}$$

where $\nu$ is a positive coefficient (viscosity) and $g$ means a nearly constant gravitational force per unit mass (an externally applied force).

Taking from Casuso, 2007 [7], the functional form derived for the velocity of a fluid

$$u_x = -\sum_k \frac{\sin(x_k t)}{it^2}\, e^{it(x+k)} + \text{const}, \tag{2}$$

where $-x_k \leqslant x + k \leqslant x_k$, $k$ describe the central positions of real matter structures such as atomic nuclei and $x_k$ means the size of these structures. Assuming a polytropic relation between pressure $P$ and density $\rho$ via the sound speed $s$ we have:

$$P = s^2\rho = \frac{s^2}{\pi}\sum_k \int \frac{\sin(x_k t)}{t}\, e^{it(x+k)} dt\,. \tag{3}$$

Puting equations (2) and (3) into equation (1) we obtain:

$$A + B = C + g\,, \tag{4}$$

where

$$A = -\sum_k \left[\frac{\cos(x_k t)}{it^2}\, x_k + \frac{(x+k)}{t^2}\sin(x_k t) + 2\,\frac{\sin(x_k t)}{t^3}\right] e^{it(x+k)}\,, \tag{5}$$

$$B = \left[-\sum_k \frac{\sin(x_k t)}{it^2}\, e^{it(x+k)} + \text{const}\right] \times \left[-\sum_k \frac{\sin(x_k t)}{t}\, e^{it(x+k)}\right], \tag{6}$$

$$C = \nu\left[-\sum_k i\sin(x_k t)\, e^{it(x+k)}\right] - \frac{is^2}{\pi}\sum_k \int \sin(x_k t)\, e^{it(x+k)}\, dt\,. \tag{7}$$

Now taking the asymptotic approximation, at very large time $t$, we obtain

$$\nu \sin(x_k t)\, e^{it(x+k)} = -\frac{s^2}{\pi}\int \sin(x_k t)\, e^{it(x+k)} dt + g\,, \tag{8}$$

and differentiating and taking only the real part, we have

$$x_k \cos(x_k t) = -\frac{s^2}{\pi\nu}\sin(x_k t)\,, \tag{9}$$

which is the same as

$$-\frac{x_k \pi\nu}{s^2} = \tan(x_k t) \tag{10}$$

then, in the limiting case (real case) $x_k \to 0$ and, again at very large time $t$, we have the solutions

$$x_k t = 0, \pi, 2\pi, 3\pi, \ldots, n\pi \tag{11}$$

with $n$ being any integer number. So we have demonstrated that the equation (2) is a solution for the Navier-Stokes equation in one dimension.

Now, for the general case of 3-dimensions we have to generalize the functional form which describes the nature of matter in Casuso, 2007 [7], in the sense of taking a new form for the density

$$\rho = \frac{1}{\pi} \sum_k \int \frac{\sin(r_k t)}{t}\, e^{it(r+k)}\, dt\,, \tag{12}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, and applying the continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u_x) - \frac{\partial}{\partial y}(\rho u_y) - \frac{\partial}{\partial z}(\rho u_z). \tag{13}$$

Using the condition of incompressibility included in Navier-Stokes equations

$$\mathrm{div}\vec{u} = 0 \tag{14}$$

and assuming isotropy for the velocity field $u_x \simeq u_y \simeq u_z$, we have

$$u_x = u_y = u_z = -\frac{r}{\pi(x+y+z)} \times \sum_k \frac{\sin(r_k t)}{it^2}\, e^{i\,t(r+k)} + \mathrm{const}, \tag{15}$$

where $-r_k \leqslant r + k \leqslant r_k$. Including this expression for the velocity in the 3-dimensional Navier-Stokes main equation (taking into account the condition $\mathrm{div}\vec{u} = 0$)

$$\frac{\partial}{\partial t} u_x = \nu \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] u_x - \frac{\partial P}{\partial x} + g\,, \tag{16}$$

we obtain

$$-\frac{r}{\pi(x+y+z)} \sum_k e^{it(r+k)} \times \left[\frac{r_k \cos(r_k t)}{i\,t^2} + \frac{(r+k)\sin(r_k t)}{t^2} - \frac{2\sin(r_k t)}{i\,t^3}\right] = = \nu \Delta u_x - \frac{\partial P}{\partial x} + g\,, \tag{17}$$

where $\Delta$ means $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Again taking the approximation of very large time, we have

$$\frac{\partial P}{\partial x} = g\,, \tag{18}$$

i.e.

$$i\,\frac{s^2 x}{\pi r} \sum_k \int \sin(r_k t)\, e^{it(r+k)} dt = g\,. \tag{19}$$

Taking the partial derivative with respect to time we obtain

$$i\,\frac{s^2 x}{\pi r} \sum_k \sin(r_k t)\, e^{it(r+k)} = 0 \tag{20}$$

or (which is the same),

$$e^{it(r+k)} \sin(r_k t) = 0\,, \tag{21}$$

i.e.

$$(\cos[(r+k)t] - i\sin[(r+k)t])\sin(r_k t) = 0\,. \tag{22}$$

Taking only the real part

$$\sin(r_k t)\cos[(r+k)t] = 0\,. \tag{23}$$

So, we have two solutions: (a) $r_k t = 0, \pi, 2\pi, \ldots, n\pi$, and (b) $(r+k)\,t = \frac{\pi}{2}, 3\frac{\pi}{2}, \ldots, (2n+1)\frac{\pi}{2}$. We must note that the solution (a) is similar to the 1-dimension solution.

## 3 Conclusions

By using a new discontinuous functional form for matter density distribution, derived from consideration of the origin of turbulence, we have found an asymptotic solution to the Navier-Stokes equation for the three dimensional case. This result, while of intrinsic interest, may point towards new ways of deriving a general solution.


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