Maxwell-Cattaneo Heat Convection and Thermal Stresses Responses of a Semi-infinite Medium due to High Speed Laser Heating

Ibrahim A. Abdallah

Department of Mathematics, Helwan University, Ain Helwan, 11795, Egypt.

E-mail: iaawavelets@yahoo.com

Based on Maxwell-Cattaneo convection equation, the thermoelasticity problem is investigated in this paper. The analytic solution of a boundary value problem for a semi-infinite medium with traction free surface heated by a high-speed laser-pulses have Dirac temporal profile is solved. The temperature, the displacement and the stresses distributions are obtained analytically using the Laplace transformation, and discussed at small time duration of the laser pulses. A numerical study for Cu as a target is performed. The results are presented graphically. The obtained results indicate that the small time duration of the laser pulses has no effect on the finite velocity of the heat conductivity, but the behavior of the stress and the displacement distribution are affected due to the pulsed heating process and due to the structure of the governing equations.

1 Introduction

The induced thermoelastic waves in the material as a response to the pulsed laser heating becomes of great interest due to its wide applications inwelding, cutting, drilling surface hardening and machining of brittle materials. The classical linear theory of thermoelasticity [1] based on Fourier relation

$$q = -k\,\frac{\partial T}{\partial x} \tag{1}$$

together with the energy conservation produces the parabolic heat conduction equation;

$$\frac{\partial T}{\partial t} = \frac{k}{c}\frac{\partial T}{\partial x^2}\,. \tag{2}$$

Although this model solved some problems on the macroscale where the length and time scales are relatively large, but it have been proved to be unsuccessful in the microscales ($< 10^{-12}$ s) applications involving high heating rates by a short-pulse laser because Fourier's model implies an infinite speed for heat propagation and infinite thermal flux on the boundaries. To circumvent the deficiencies of Fourier's law in describing such problems involving high rate of temperature change; the concept of wave nature of heat transformation had been introduced [2, 3]. Beside the coupled thermoelasticity theory formulated by Biot [4], thermoelasticity theory with one relaxation time introduced by Lord and Shulman [5] and the two-temperature theory of thermoelasticity [6] which introduced to improve the classical thermoelasticity, there is the Maxwell-Cattaneo model of heat convection [9].

In the Maxwell-Cattaneo model the linkage between the heat conduction equation

$$q + \tau\,\frac{\partial q}{\partial t} = -k\,\frac{\partial T}{\partial x} \tag{3}$$

and the energy conservation introduces the hyperbolic equation

$$\tau\,\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \frac{k}{c}\frac{\partial^2 T}{\partial x^2} \tag{4}$$

which describes a heat propagation with finite speed. The finiteness of heat propagation speed provided by the generalized thermoelasticity theories based on Maxwell- Cattaneo model of convection are supposed to be more realistic than the conventional theory to deal with practical problems with very large heat fluxes and/or short time duration.

Biot [4] formulated the theory of coupled thermoelasticity to eliminate the shortcoming of the classical uncoupled theory. In this theory, the equation of motion is a hyperbolic partial differential equation while the equation of energy is parabolic. Thermal disturbances of a hyperbolic nature have been derived using various approaches. Most of these approaches are based on the general notion of relaxing the heat flux in the classical Fourier heat conduction equation, thereby, introducing a non Fourier effect.

The first theory, known as theory of generalized thermoelasticity with one relaxation time, was introduced by Lord and Shulman [5] for the special case of an isotropic body. The extension of this theory to include the case of anisotropic body was developed by Dhaliwal and Sherief [7]. Recently, the author and co-workers investigated the problem of thermoelasticity, based on the theory of Lord and Shulman with one relaxation time, is used to solve a boundary value problem of one dimensional semiinfinite medium heated by a laser beam having a temporal Dirac distribution [8].

The purpose of the present work is to study the thermoelastic interaction caused by heating a homogeneous and isotropic thermoelastic semi-infinite body induced by a Dirac pulse having a homogeneous infinite cross-section by employing the theory of thermoelasticity with one relaxation time. The problem is solved by using the Laplace transform technique. Approximate small time analytical solutions to stress, displacement and temperature are obtained. The convolution theorem is applied to get the spatial and temporal temperature distribution induced by laser radiation having a temporal Gaussian distribution. At the end of this work a numerical study for Cu as a target is performed and presented graphically and concluding remarks are given.

2 Formulation of the problem

We consider one-dimensional heating situation thermoelastic, homogeneous, isotropic semi-infinite target occupying the region $z \geqslant 0$, and initially at uniform temperature T_0. The surface of the target $z = 0$ is heated homogeneously by a leaser beam and assumed to be traction free. The Cartesian coordinates (x, y, z) are considered in the solution and z-axis pointing vertically into the medium. The governing equations are: The equation of motion in the absence of body forces

$$\sigma_{ji,j} = \rho\, \ddot{u}_i\,, \qquad i, j = x, y, z \tag{5}$$

where σ_{ij} is the components of stress tensor, u_i's are the displacement vector components and ρ is the mass density.

The Maxwell-Cattaneo convection equation

$$\frac{\partial\theta}{\partial t} + \tau\,\frac{\partial^2\theta}{\partial t^2} = \frac{k}{\rho c_E}\frac{\partial^2\theta}{\partial z^2} \tag{6}$$

where c_E is the specific heat at constant strain, τ is the relaxation time and k is the thermal conductivity.

The constitutive equation

$$\sigma_{ij} = (\lambda\,\mathrm{div}\,u - \gamma\theta)\,\delta_{ij} + 2\mu\,\epsilon_{ij} \tag{7}$$

where δ_{ij} is the delta Kronecker, $\gamma = \alpha_t(3\lambda + 2\mu)$, λ, μ are Lame's constants and α is the thermal expansion coefficient.

The strain-displacement relation

$$\epsilon_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i})\,, \qquad i, j = x, y, z \tag{8}$$

The boundary conditions:

$$\sigma_{zz} = 0\,, \quad \text{at} \quad z = 0\,, \tag{9}$$

$$-k\,\frac{d\theta}{dz} = A_0\,q_0\,\delta(t)\,, \quad \text{at} \quad z = 0\,, \tag{10}$$

$$\sigma_{zz} = 0\,, \quad w = 0\,, \quad \theta = 0\,, \quad \text{as} \quad z \to \infty\,, \tag{11}$$

where A_0 is an absorption coefficient of the material, q_0 is the intensity of the laser beam and $\delta(t)$ is the Dirac delta function [10]. The initial conditions:

$$\left.\begin{aligned} &\theta(z,0) = \theta_0\,, \quad w(z,0) = 0\,, \quad \sigma_{ij}(z,0) = 0 \\ &\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial t^2} = \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial t^2} = \frac{\partial\sigma_{ij}}{\partial t} = \frac{\partial^2\sigma_{ij}}{\partial t^2} = 0 \\ &\qquad\qquad\qquad\qquad\qquad \text{at} \quad t = 0\,, \quad \forall z \end{aligned}\right\}. \tag{12}$$

Due to the symmetry of the problem and the external applied thermal field, the displacement vector u has the components:

$$u_x = 0\,, \quad u_y = 0\,, \quad u_z = w(z,t)\,. \tag{13}$$

From equation (12) the strain components ϵ_{ij}, read;

$$\left.\begin{aligned} &\epsilon_{xx} = \epsilon_{yy} = \epsilon_{xy} = \epsilon_{xz} = \epsilon_{yz} = 0 \\ &\epsilon_{zz} = \frac{\partial w}{\partial z} \\ &\epsilon_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i})\,, \qquad i, j = x, y, z \end{aligned}\right\}. \tag{14}$$

The volume dilation e takes the form

$$e = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = \frac{\partial w}{\partial z}\,. \tag{15}$$

The stress components in (8) can be written as:

$$\left.\begin{aligned} &\sigma_{xx} = \sigma_{yy} = \lambda\,\frac{\partial w}{\partial z} - \gamma\theta \\ &\sigma_{zz} = (2\mu + \lambda)\,\frac{\partial w}{\partial z} - \gamma\theta \end{aligned}\right\}, \tag{16}$$

where

$$\left.\begin{aligned} \sigma_{xy} &= 0 \\ \sigma_{xz} &= 0 \\ \sigma_{yz} &= 0 \end{aligned}\right\}. \tag{17}$$

The equation of motion (5) will be reduce to

$$\sigma_{xz,x} + \sigma_{yz,y} + \sigma_{zz,z} = \rho\ddot{u}_z\,. \tag{18}$$

Substituting from the constitutive equation (8) into the above equation and using $\theta = T - T_0$ we get,

$$(2\mu + \lambda)\,\frac{\partial^2 w}{\partial z^2} - \gamma\,\frac{\partial\theta}{\partial z} = \rho\,\frac{\partial^2 w}{\partial t^2} \tag{19}$$

where θ is the temperature change above a reference temperature T_0. Differentiating (19) with respect to z and using (15), we obtain

$$(2\mu + \lambda)\,\frac{\partial^2 e}{\partial z^2} - \gamma\,\frac{\partial^2\theta}{\partial z^2} = \rho\,\frac{\partial^2 e}{\partial t^2} \tag{20}$$

after using (6) the energy equation can be written in the form:

$$(2\mu + \lambda)\,\frac{\partial^2 e}{\partial z^2} - \rho\,\frac{\partial^2 e}{\partial t^2} = \frac{\gamma\rho\,c_E}{k}\left(\frac{\partial}{\partial t} + \tau\,\frac{\partial^2}{\partial t^2}\right)\theta \tag{21}$$

by this equation one can determine the dilatation function e after determining θ which can be obtained by solving (6) using Laplace transformation; $\bar{f}(z,s) = \int_0^\infty e^{-st} f(z,t)\,dt$.

3 Analytic solution

In this section we introduce the analytical solutions of the system of equations (6), (16) and (19) based on the Laplace transformation. Equation (6) after applying the Laplace transformation it will be;

$$\frac{d^2\bar{\theta}}{dz^2} - \alpha s\,(1+\tau s)\,\bar{\theta} = 0 \tag{22}$$

where $\alpha = \frac{\rho c_E}{k}$. By solving the above equation and using the boundary and the initial conditions (9)-(12); one can write the solution of equation (22) as

$$\bar{\theta} = \frac{A_0 q_0}{k\,f(s)} e^{-f(s)z}, \qquad \mathrm{Re}\,(f(s)) > 0\,. \tag{23}$$

Similarly the solution of equation (19) after Laplace transformation read;

$$\bar{w}(z,s) = B(s)\,e^{-asz} - \frac{\beta}{(f^2(s) - a^2 s^2)}\,e^{-f(s)z} \tag{24}$$

where

$$a^2 = \frac{\rho}{(2\mu+\lambda)}\,, \quad f(s) = \sqrt{\alpha s\,(1+\tau s)}\,, \quad \beta = \frac{\gamma A_0 q_0}{k(2\mu+\lambda)}\,,$$

$$B(s) = \frac{\beta f(s)}{sa\,(2\mu+\lambda)\,(f^2(s)-a^2s^2)} - \frac{\beta}{as f(s)}\,.$$

Since we can use the Maclaurin series to write

$$\sqrt{s(1+\tau s)} = \sqrt{s^2\left(\tau + \frac{1}{s}\right)} \approx s\sqrt{\tau} + \frac{1}{2\sqrt{\tau}}\,. \tag{25}$$

Then the solution of the temperature distribution $\bar{\theta}$, and the displacement $\bar{w}$ can be written as

$$\bar{\theta}(z,s) = \left[\frac{C_1}{s} - \frac{C_2}{s^2} + \frac{C_3}{s^3}\right] e^{-z(s\sqrt{\alpha\tau}+\frac{1}{2}\sqrt{\frac{\alpha}{\tau}})}\,, \tag{26}$$

$$\begin{aligned}\bar{w}(z,s) = &\left[\frac{w_1}{s} + \frac{w_2}{\frac{\alpha}{b}+s} + \frac{w_3}{s+\frac{1}{\tau}}\right] e^{-asz} - \\ &- \left[\frac{w_4}{s} + \frac{w_5}{\frac{\alpha}{b}+s}\right] e^{-z(s\sqrt{\alpha\tau}+\frac{1}{2}\sqrt{\frac{\alpha}{\tau}})}\,,\end{aligned} \tag{27}$$

therefore the stresses $\bar{\sigma}_{zz}$ and $\bar{\sigma}_{xx} = \bar{\sigma}_{yy}$ are obtained by applying the Laplace transformation to equation (16) and substituting by (26) and (27). Then using the inverse Laplace transformation, we obtain: the temperature θ

$$\begin{aligned}\theta(z,t) = &\Big[C_1 - C_2(t - \sqrt{\alpha\tau}z) + \\ &+ \frac{C_3}{2}(t - \sqrt{\alpha\tau}z)^2\Big] H(t - \sqrt{\alpha\tau}z) e^{-\frac{z}{2}\sqrt{\frac{\alpha}{\tau}}}\,,\end{aligned} \tag{28}$$

the displacement w

$$\begin{aligned}w(z,t) = &\left[w_1 + \frac{w_2}{b}\,e^{-\frac{\alpha}{b}(t-az)} + w_3\,e^{-\frac{t-az}{\tau}}\right] H(t-az) - \\ &- \left[w_4 + \frac{w_5}{b}\,e^{-\frac{\alpha}{b}(t-\sqrt{\alpha\tau}z)}\right] H(t-\sqrt{\alpha\tau}z)\,e^{-\sqrt{\frac{\alpha}{\tau}}z}\,,\end{aligned} \tag{29}$$

the stresses $\sigma_{xx} = \sigma_{yy}$

$$\begin{aligned}\sigma_{xx}(z,t) = &-a\lambda\Big[L_1\delta(t-az) - L_2H(t-az)\,e^{-\frac{\alpha}{b}(t-az)} - \\ &- L_3H(t-az)e^{-\frac{1}{\tau}(t-az)}\Big] + \\ &+ e^{-\frac{z}{2}\sqrt{\frac{\alpha}{\tau}}}\Big[L_4\delta(t-\sqrt{\alpha\tau}) + H(t-\sqrt{\alpha\tau}z)\,\times \\ &\times (L_5 + L_6 e^{-\frac{\alpha}{b}(t-\sqrt{\alpha\tau}z)})\Big] - \gamma\Big[C_1 - C_2(t-\sqrt{\alpha\tau}z) + \\ &+ \frac{C_3}{2}(t-\sqrt{\alpha\tau}z)^2\Big] H(t-\sqrt{\alpha\tau}z)\,e^{-\frac{z}{2}\sqrt{\frac{\alpha}{\tau}}}\,,\end{aligned} \tag{30}$$

the stress σ_{zz}

$$\begin{aligned}\sigma_{zz}(z,t) = &-a(2\mu+\lambda)\Big[L_1\delta(t-az) - \\ &- L_2H(t-az)\,e^{-\frac{\alpha}{b}(t-az)} - L_3H(t-az)\,e^{-\frac{1}{\tau}(t-az)}\Big] + \\ &+ e^{-\frac{z}{2}\sqrt{\frac{\alpha}{\tau}}}\Big[L_4\delta(t-\sqrt{\alpha\tau}) + H(t-\sqrt{\alpha\tau}z)\,\times \\ &\times (L_5 + L_6\,e^{-\frac{\alpha}{b}(t-\sqrt{\alpha\tau}z)})\Big] - \gamma\Big[C_1 - C_2(t-\sqrt{\alpha\tau}z) + \\ &+ \frac{C_3}{2}(t-\sqrt{\alpha\tau}z)^2\Big] H(t-\sqrt{\alpha\tau}z)\,e^{-\frac{z}{2}\sqrt{\frac{\alpha}{\tau}}}\,,\end{aligned} \tag{31}$$

$\delta(x)$ is Dirac delta function, and $H(x)$ is Heaviside unit step functions.

4 Results and discussions

We have calculated the spatial temperature, displacement and stress θ, w, σ_{xx}, σ_{yy} and σ_{zz} with the time as a parameter for a heated target with a spatial homogeneous laser radiation having a temporally Dirac distributed intensity with a width of (10^{-3} s). We have performed the computation for the physical parameters $T_0 = 293$ K, $\rho = 8954$ Kg/m^3, $A_0 = 0.01$, $c_E = 383.1$ J/kgK, $\alpha_t = 1.78\times10^{-5}$ K^{-1}, $k = 386$ W/mK, $\lambda = 7.76\times10^{10}$ kg/m sec^2, $\mu = 3.86\times10^{10}$ kg/m sec^2 and $\tau = 0.02$ sec for Cu as a target. Therefore the coefficients in the expressions (28)–(31)are

$$\left.\begin{array}{ll} C_1 = 1676.0\,, & C_2 = -83800.2 \\ C_3 = 1.57125\times10^6 & \\ w_1 = -5760.28\,, & w_2 = 44906.0 \\ \frac{w_2}{b} = 63506.0\,, & w_3 = 1.5589\times10^6 \\ w_4 = 0.1039\,, & w_5 = -0.7348 \\ \frac{\alpha}{b} = 1256.77\,, & L_1 = -3.0172\times10^{13} \\ L_2 = 1.4896\times10^5, & L_3 = 1.4547\times10^5 \\ L_4 = 2.7708\,, & L_5 = 34.6344 \\ L_6 = 1.7065\times10^3, & \frac{w_5}{b} = 0.103916 \end{array}\right\}. \tag{32}$$

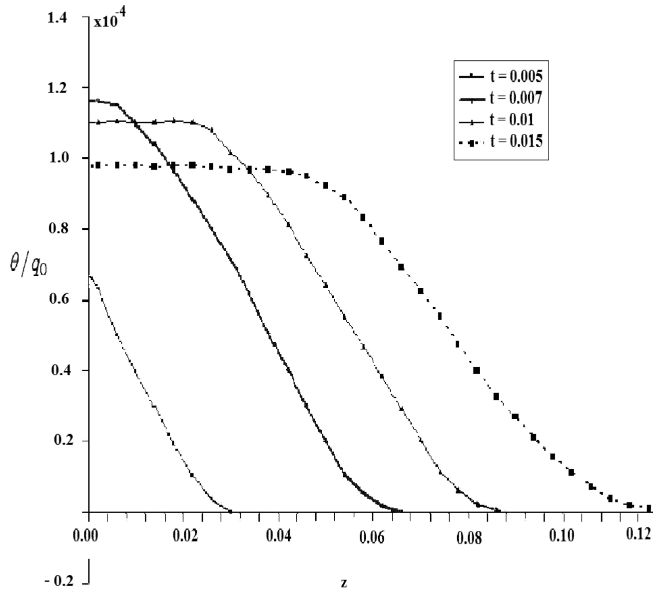

Fig. 1: The temperature distribution θ per unit intensity versus z with the time as a parameter.

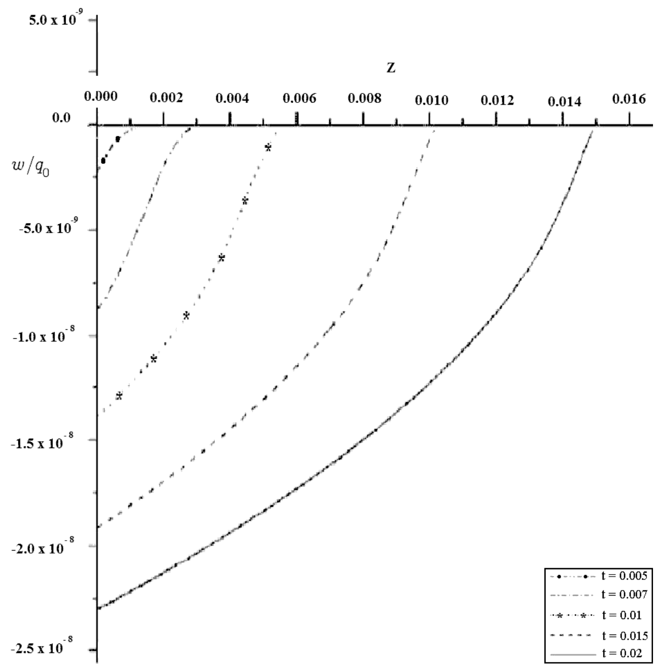

Fig. 2: The displacement distribution w per unit intensity versus z at different values of time as a parameter.

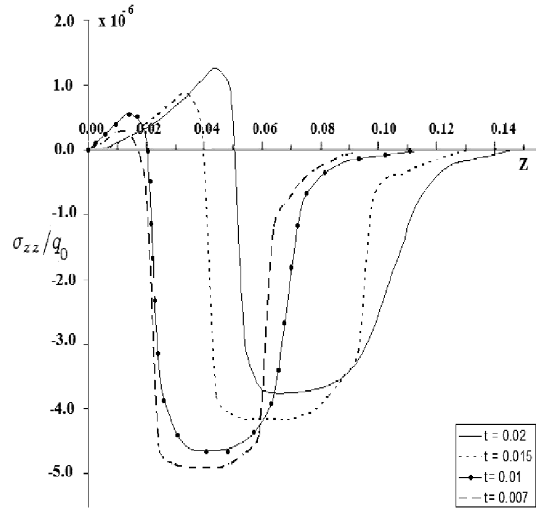

Fig. 3: The stress σ_{zz} distribution per unit intensity versus z with the time as a parameter.

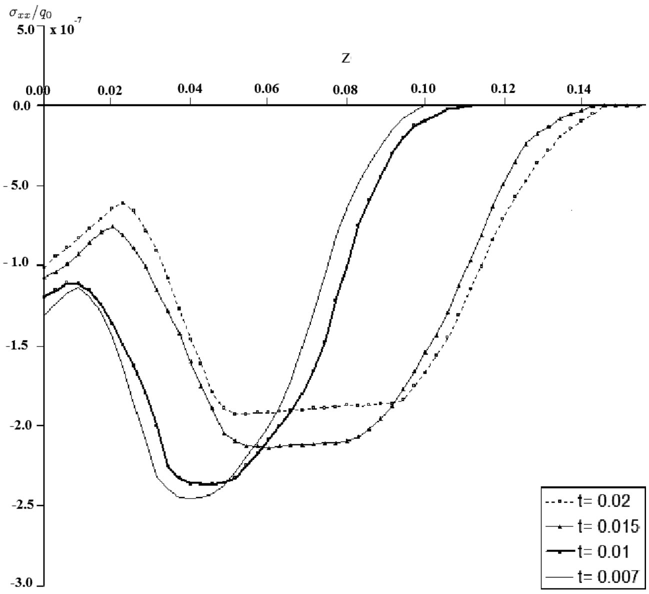

Fig. 4: The stress distribution $\sigma_{xx} = \sigma_{yy}$ per unit intensity versus z with the time as a parameter.

The obtained results are shown in the following figures.

Figure 1 illustrates the calculated spatial temperature distribution per unit intensity at different values of the time as a parameter $t = 0.005$, 0.007, 0.01 and 0.015. From the curves it is evident that the temperature has a finite velocity expressed through the strong gradient of the temperature at different locations which moves deeper in the target as the time increases.

Figure 2 represents the calculated spatial displacement per unit intensity for different values of time as a parameter. The displacement increases monotonically with increasing z. It Shows a smaller gradient with increasing z this behavior occurs at smaller z values than that of the temperature calculated at the corresponding time when it tends to zero. Both effects can be attributed to the temperature behavior and the finite velocity of the expansion which is smaller than that of the heat conduction. The negative displacement indicate the direction of the material expandion where the co-ordinate system is located at the front surface with positive direction of the $z-$axis pointing in the semi-infinite medium.

Figure 3 shows the calculated spatial stress σ_{zz} per unit intensity calculated at different times. It is given by $\sigma_{zz} = = \alpha e - \lambda_1 \theta$. For small z values and at the time $t = 0.005$ the temperature attains greater values than the gradient of the displacement, thus the stress in z direction becomes negative. After attaining z greater values both the temperature and the gradient of the displacement become smaller such that σ_{zz} takes greater values tending to zero. For $t = 0.007$ the effect of the temperature is dominant more than that of the gradient of the displacement this is leading to a more negative stress values shifted toward greater values of z. As the value $t = 0.01$ the effect of the gradient of the displacement over compensates that of the temperature leading to positive stress values lasting up to locations at which the gradient of the displacement and the temperature are practically equal. At this point the stress becomes maximum. As z takes greater values the gradient of the displacement decreases and the temperature becomes the upper hand leading to negative stress values. These behavior remains up to z values at which the temperature is practically zero where the stress tends also be zero. As t takes greater values the effect of the gradient will by more pronounced and thus the maximum of the stress becomes greater and shifts towards the greater z values.

Figure 4 depicts the calculated spatial stress distributions $\sigma_{xx} = \sigma_{yy}$ per unit intensity at different values of the time parameter. the same behavior as σ_{zz}. This is due to the same dependent relation of σ_{ij} on the strain and temperature except that the coefficient of the strain is different.

5 Conclusions

The thermoelastic waves in a semi infinite solid material indused by a Dirac pulsed laser heating are derived for non-Fourier effect based on the Maxwell-Cattaneo hyperbolic convection equation. Analytical solution for the temperature, the displacement and the stresses fields inside the material are derived using the Laplace transformation. The carried calculations enable us to model the thrmoelastic waves induced by a high speed Dirac laser pulse. From the figures it is evident that the temperature firstly increases with increasing the time this can be attributed to the increased absorbed energy which over compensates the heat losses given by the heat conductivity inside the material. As the absorbed power equals the conducted one inside the material the temperature attains its maximum value. the maximum of the temperature occurs at later time than the maximum of the radiation this is the result of the heat conductivity of Cu and the relatively small gradient of the temperature in the vicinity of $z = 0$. After the radiation becomes week enough such that it can not compensate the diffused power inside the material the temperature decreases monotonically with increasing time. Considering surface absorption the obtained results in Figure 1 shows the temperature θ, Figure 2 shows the displacement w, Figure 3 shows the the stress σ_{zz}, and Figure 4 shows the stresses $\sigma_{xx} = \sigma_{yy}$ respectively versus z. The solution of any of the considered function for this model vanishes identically to zero outside a bounded region. The response to the thermal effects by pulsed Laser heating does not reach infinity instantaneously but remains in a bounded region of z given by $0 < z < z^*(t)$ where t is the duration of the laser pulse used for heating. The stress exhibits like step-wise changes at the wave front. The stresses vanish quickly due to the dissipation of the thermal waves.

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