

Primes, Geometry and Condensed Matter

Riadh H. Al Rabeh

University of Basra, Basra, Iraq

E-mail: alrabeh_rh@yahoo.com

Fascination with primes dates back to the Greeks and before. Primes are named by some “the elementary particles of arithmetic” as every nonprime integer is made of a unique set of primes. In this article we point to new connections between primes, geometry and physics which show that primes could be called “the elementary particles of physics” too. This study considers the problem of closely packing similar circles/spheres in 2D/3D space. This is in effect a discretization process of space and the allowable number in a pack is found to lead to some unexpected cases of prime configurations which is independent of the size of the constituents. We next suggest that a non-prime can be considered geometrically as a symmetric collection that is separable (factorable) into similar parts- six is two threes or three twos for example. A collection that has no such symmetry is a prime. As a result, a physical prime aggregate is more difficult to split symmetrically resulting in an inherent stability. This “number/physical” stability idea applies to bigger collections made from smaller (prime) units leading to larger stable prime structures in a limitless scaling up process. The distribution of primes among numbers can be understood better using the packing ideas described here and we further suggest that differing numbers (and values) of distinct prime factors making a nonprime collection is an important factor in determining the probability and method of possible and subsequent disintegration. Disintegration is bound by energy conservation and is closely related to symmetry by Noether theorems. Thinking of condensed matter as the packing of identical elements, we examine plots of the masses of chemical elements of the periodic table, and also those of the elementary particles of physics, and show that prime packing rules seem to play a role in the make up of matter. The plots show convincingly that the growth of prime numbers and that of the masses of chemical elements and of elementary particles do follow the same trend indeed.

1 Introduction

Primes have been a source of fascination for a long time- as far back as the Greeks and much before. One reason for this fascination is the fact that every non-prime is the product of a unique set of prime numbers, hence the name *elementary particles of arithmetic*, and that although primes are distributed seemingly randomly among other integers, they do have regular not fully understood patterns (see [1] for example). The literature is rich in theories on primes but one could say that none-to-date have managed to make the strong connection between primes and physics that is intuitively felt by many. One recent attempt in this direction is [2], wherein possible connections between the atomic structure and the zeros of the Zeta function — closely connected to primes — are investigated. We quote from this reference, “Why the periodicity of zeros from the Riemann-Zeta function would match the spacing of energy levels in high-Z nuclei still remains a mystery”.

In the present work we attempt to relate primes to both geometry and physics. We start with the packing of circles in a plane (or balls on a plane)- all of the same size, and pose a question; *In a plane, what is the condition for packing an integral number of identical circles to form a larger circle- such that both the diameter and circumference of the larger circle contain an integral numbers of the small circle?* The problem

is essentially the same when the 2D circles are replaced with balls on a tray. A surprising result here is the appearance of only two prime numbers 2 and 3 in the answer and *only one of them is nontrivial- the number 3*. This gives such numbers a fundamental and natural importance in geometry. We may view this number as a “discretization number of the continuous 3D spaces”. We further study this matter and shed light (using balls to represent integers) on bounds on the growth of primes- namely the well known logarithmic law in the theory of primes. Still further, we coin the notion that distinct prime factors in the packing of composite collections/grouping can have a profound influence on the behaviour of such collections and the manner they react with other collections built of some different or similar prime factors. As many physics models of condensed matter assume identical elements for simple matter (photons, boson and fermion statistics and the MIT bag model [3, 6] are examples) we examine the applicability of our packing rules in such case and conclude that condensed matter do seem to follow the packing rules discussed here.

2 Theory

Consider the case of close packing of circles on a plane so as to make a bigger circle (Figure 1). The ratio of the radius of the large circle to that of the small circle is; $R/r = 1 + 1/\sin t$,

1	7	13	19	25	31	37	43	49	55	61	67	73	79	85	91	97...
2	8	14	20	26	32	38	44	50	56	62	68	74	80	86	92	98...
3	9	15	21	27	33	39	45	51	57	63	69	75	81	87	93	99...
4	10	16	22	28	34	40	46	52	58	64	70	76	82	88	94	100...
5	11	17	23	29	35	41	47	53	59	65	71	77	83	89	95	101...
6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102...

Table 1: Integers arranged in columns of six.

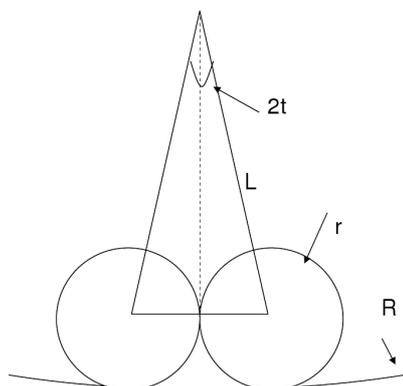


Fig. 1: Close packing of an integral number of circles/balls on a plane have one nontrivial solution- 6 balls, plus one at the centre (see also Figure 2). Here in Fig. 1: $L \sin t = r$; $t = \pi/n$; $R = L + r$; $R/r = 1 + 1/\sin t$. For integral ratio R/r , t must be $\pi/2$ or $\pi/6$ and $L/r = 2, 3$.

where t is half the angle between radial lines through the centers of any two adjacent circles. For this number to be an integer, the quantity $(1/\sin t)$ must be an integer and hence the angle t must be either 30 or 90 degrees. Thus R/r should be either three or two (see Figure 2b). That is; the diameter can be either two or three circles wide. The number 3 is non-trivial, and gives six circles touching each other, and all in turn tangent to a seventh circle at the centre.

Clearly the arrangement of balls on a plane does follow exactly the same pattern leading to six balls touching in pairs and surrounding a seventh ball (touching all other six) at the center. This result is unique and is independent of the size of the balls involved. It is rather remarkable as it gives the number 6 a special stature in the physics of our 3D space, parallel to that of the number π in geometry. Such stature must have been realized in the past by thinkers as far back as the Babylonian times and the divine stature given to such a number in the cultures of many early civilizations- six working days in a week and one for rest is one example, the six prongs of the star of David and the seven days of creation as well as counting in dozens might have also been inspired by the same. Before this, the Bees have discovered the same fact and started building their six sided honey combs accordingly.

Consider now the set of prime numbers. It is known that every prime can be written as $6n \pm 1$, where n is an integer. That is the number six is a generator of all primes. Further, we

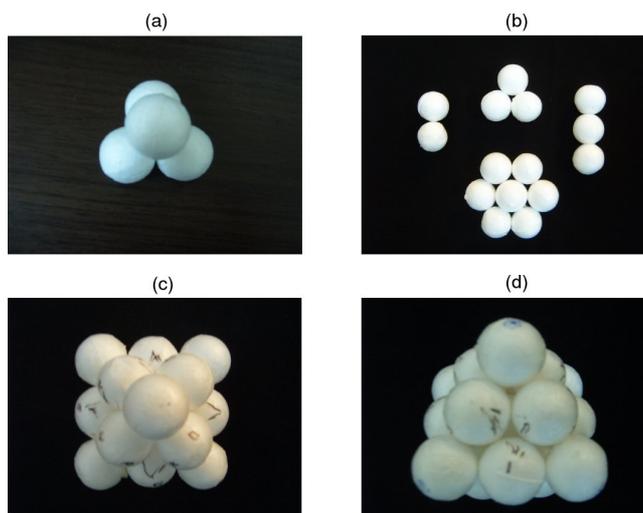


Fig. 2: Packing of 2, 3, 4 & 19 ($=7+(3+3)+(3+3)$) balls in 3D (a, b). The 19 ball case possesses six side and eight side symmetries (c, d).

note that whereas the number six is divisible into 2 (threes) or 3 (twos), an addition of one unit raises the number to seven- a prime and not divisible into any smaller symmetric entities. Put differently, an object composed of six elements can easily break into smaller symmetrical parts, whereas an object made of 7 is more stable and not easily breakable into symmetric parts. We know from physics that symmetry in interactions is demanded by many conservation laws. In fact symmetry and conservation are tightly linked by Noether theorems- such that symmetry can always be translated to a conservation law and vice versa. When we have a group of highly symmetric identical items, the addition of one at the centre of the collection can make it a prime.

Now if we arrange natural numbers in columns of six as shown in Table 1, we see clearly that all primes fall along two lines- top line for the $6n + 1$ type and the bottom line for the case of $6n - 1$ type primes (text in bold). If these are balls arranged physically on discs six each and on top of each other, the two lines will appear diametrically opposite on a long cylinder. Thus there are two favourite lines along which all primes fall in a clear display of a sign of the close connection between primes geometry and physics.

We see then that the connection between primes and geometry is an outcome of how the plane and the space lend themselves to discretization, when we pair such blocks with

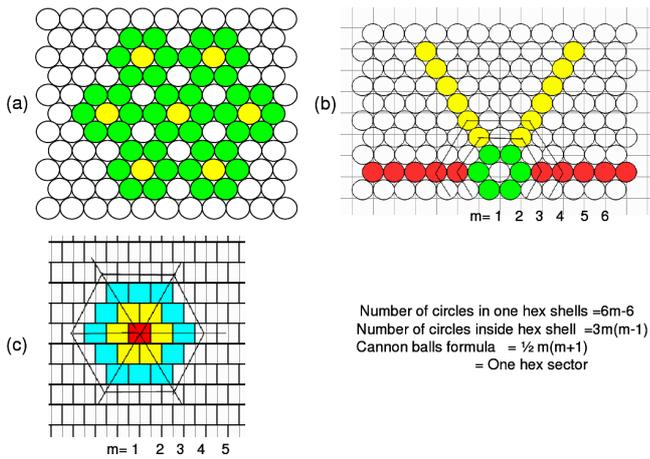


Fig. 3: (a) Scaling up using small blocks of seven to make larger blocks of seven; (b) Tight packing of circles naturally resulting in hex objects made of hex layers. The number of circles in each layer strip increases in steps of 6. Note that each hex sector has cannon balls (or conical) packing structure; (c) Easy to construct (square) brick structure to formally replace circles.

the set of positive integers. We may note also that the density of $6n - 1$ and $6n + 1$ type primes is the same with respect to the integers. Moreover, if we take the difference between prime pairs, the distribution of the difference peaks at 6 and all multiples of it, but diminishes as the difference increases (Figure 4c).

In a violent interaction between two prime groups, one or more of the groups could momentarily loose a member or more leaving a non-prime group which then become less stable and divisible into symmetric parts according to the factors making the collection. Clearly in this case, the few none primes neighbouring a prime also become important, and would contribute to the rules of break-up, to the type of products and to the energy required in each case.

Our packing endeavour can continue beyond 7 to make larger 3D objects (Figure 2). A stable new arrangement can result from the addition of 6 balls- 3 on each side (top and bottom) making an object of 13 balls- a new prime figure. Further 6 balls can be put symmetrically secured on top and bottom to give an object of 19 balls. This last case in addition to being a prime collection has an interesting shape feature. It has six and eight face symmetries and fairly smooth faces as shown in Figures 2(c, d), which could give rise to two different groups of 19 ball formations. Further addition of 6's is possible, but the resulting object appears less strong. To go a different direction, we can instead consider every 7, 13 or 19 ball objects as the new building unit and use it to form further new collections of objects of prime grouping. Clearly this can be continued in an endless scaling up process (Figure 3b). Scaling is a prominent phenomenon in physical structures. Fig. 3b shows that, in a plane, our packing problem and also that of the packing of cannon-balls [5] are only subsets of the general densest packing problem and thus it truly is a dis-

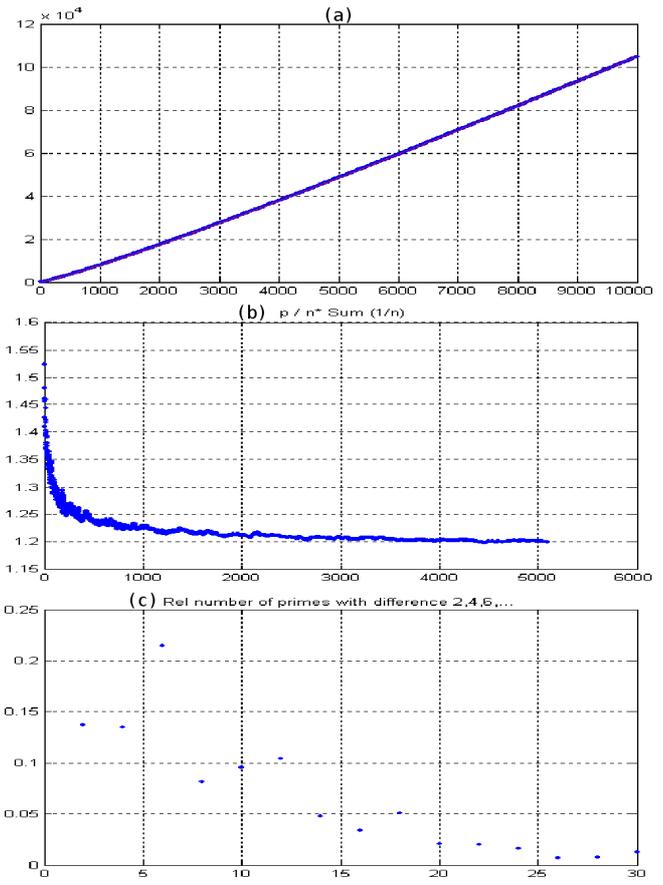


Fig. 4: (a) Two overlapping plots of the first 104 primes: (1, 2, 3, 5, ..., 104729) compared to fitting plot (-), $y = (\ln \pi) \cdot n \cdot \ln n$ ($n =$ serial positions of prime numbers) (.); (b) Ratio of a prime (p) to $n \sum 1/n$; (c) Relative number of primes with differences of 2, 4, 6, ..., 30. Peaks occur at differences of 6, 12, 18, 24, 30.

cretization process of space. We note also that circles can be replaced with squares placed in a brick like structure provided we only think of the centres of these squares.

In the process of adding new rings of circles to form larger objects, both prime and nonprime numbers are met. A prime is formed every time we have highly symmetric combination with one to be added or subtracted to it to break the symmetry and produce a prime. If we consider the number of circles added in each ring in the case of circular geometry (the same applies to hex geometry with small modification), the radius of a ring is given by $mr + m$, where m is the number of layers and r is the radius of one small circle set to unity. The number of circles in each ring is estimated by the integer part of $2\pi m$. For the next ring we substitute $(m + 1)$ for m in the above expressions and obtain $2\pi(m + 1)$ for a ring. The relative increase in the number of circles is the difference between these two divided by the circumference which gives $1/m$. The relative (or probable) number of primes for m -th ring should be taken to come from the contribution of all the items in the ring and this is proportional to $\sum 1/m$ for large m . The actual number of primes is an integral of this given by $m \sum 1/m$

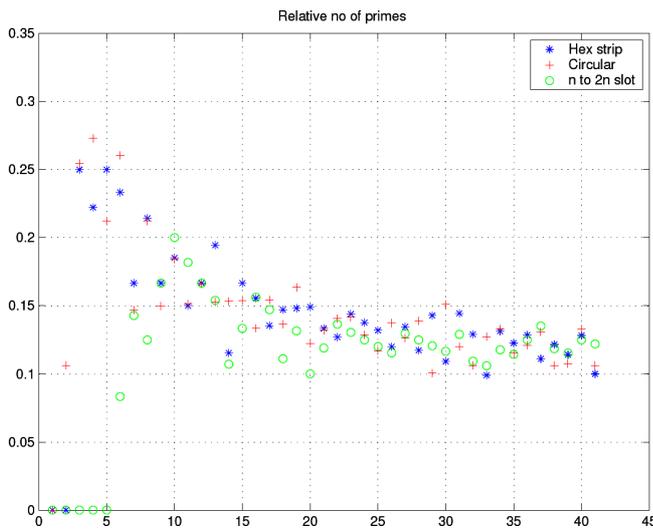


Fig. 5: Relative number of primes in (50000 integer sample): Hex strips $m : m + 1$ (*); Circular rings $m : m + 1$, m is the number of rings of circles around the centre (+); Interval $n : 2n$, n is the serial number of a prime (o).

since both the radius of a circular strip and the number of circles in that strip are proportional to m . Fig. 5 gives the relative number of primes in one strip and the trend is of the form $a / \log m$, thus confirming the reasoning used above.

Figure 4b gives a plot of the ratio $p / (n \sum 1/n)$ where p is the value of the n -th prime for some 50000 primes sample which, for large n , equals the number of integers/circles in the whole area. Since $\sum 1/n \sim \ln(n)$ for large n , we see that this ratio tends to a constant in agreement with the results of the prime number theory (see [1] for example).

Further, there are few results from the theory of primes that can also be interpreted in support of the above arguments. For example the well known conjectures suggesting that there is always a prime between m and $2m$ and also between m^2 and $(m + 1)^2$ [7] can respectively be taken to correspond to the symmetrical duplication of an area and to the ring regions between two concentric circles must contain at least one prime. That is if the original area or sector can produce a prime, then duplicating it symmetrically or adding one more sector to it will produce at least one prime. The number of primes in each of the above cases and that of a hex region are of course more than one and the results from a sample of (1–50000) integers are plotted in Figure 5. The data is generated using a simple Excel-Basic program shown below;

```
%Open excel > Tools > micro > Basic Editor > paste and run
subroutine prime( )
kk=0:
% search divisibility up to square root
for ii=1 to 1e6: z=1: iis=int(sqrt(ii+1)+1):
% test divisibility
for jj=2 to iis: if ii-int(ii/jj)*jj=0 then z=0: next jj:
% write result in excel sheet
if z=1 then kk=kk+1: if z=1 then cells(kk,1)=ii: next ii:
end sub:
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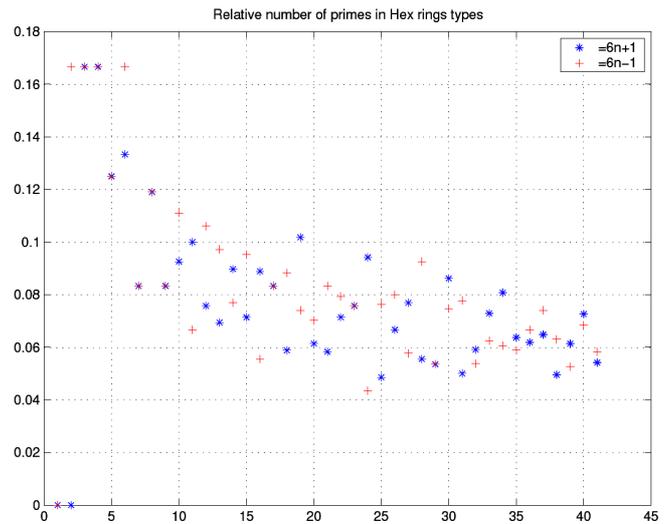


Fig. 6: Relative number of primes in hex strips (see Figure 3b); primes of the form $6n + 1$ (*); primes of the form $6n - 1$, n is the number of primes around the centre (+).

Concentric circles can be drawn on top of the hexagons shown in Figure 3, and the number of smaller circles tangent to the large circles then occur in a regular and symmetrical way when the number of circular layers is a prime. Some attempt was made by one researcher to explain this by forming and solving the associated Diophantine equations. It is noted here that potential energy and forces are determined by radial distances- that is the radii of the large circles. Also it is known that the solution of sets of Diophantine equations is a generator of primes.

None prime numbers can be written in a unique set of primes. Thus for any number P we have;

$$P = p_1^a p_2^b p_3^c \dots \quad \text{and} \\ \log P = a \log p_1 + b \log p_2 + c \log p_3 \dots$$

where a, b, c are integral powers of the prime factors $p_1 p_2 \dots p_n$. Ref. [8] have observed that this relation is equivalent to energy conservation connecting the energy of one large object to the energy of its constituents- where energy is to be associated with $(\log P)$. Further, if the values of a, b, c are unity, the group would only have one energy state (structure), and could be the equivalent of fermions in behaviour. When the exponents are not unity (integer > 1), the group would behave as bosons and would be able to exist in multiple equivalent energy states corresponding to the different combination values of the exponents. Note that $\log p$ would correspond to the derivative of the prime formula $(n \log n)$ for large n accept for a negative sign.

Still in physics, we note that the size of the nucleus of chemical elements is proportional to the number of nucleons [3, 6] inside it. Since many of the physical and statistical models of the nucleus assume identical constituents, we may think of testing the possibility of condensed matter fol-

I – Elementary particles;		II – Particle mass/electron mass;					III – Nearest primes			
I-	e	μ	π^0	π^\pm	K^\pm	K^0	η	ρ	ω	K^*
II-	1	206.7	264.7	274.5	966.7	974.5	1074.5	1506.8	1532.3	1745.5
III-	1	211	263	277	967	977	1069	1511	1531	1747
I-	p	n	η'	φ	Λ	Σ^+	Σ^0	Σ^-	Δ	Ξ^0
II-	1836.2	1838.7	1873.9	1996.1	2183.2	2327.5	2333.6	2343.1	2410.9	2573.2
III-	1831	1831	1877	2003	2179	2333	2339	2347	2417	2579
I-	Ξ^-	Σ^*	Ξ^*	Ω^-	τ	D^0	D^\pm	F^\pm	D^*	Λ_c^-
II-	2585.7	2710.4	3000	3272	3491.2	3649.7	3657.5	3857.1	3933.4	4463.8
III-	2591	2713	3001	3271	3491	3643	3671	3863	3931	4463

Table 2: Relative masses of well known elementary particles and their nearest primes.

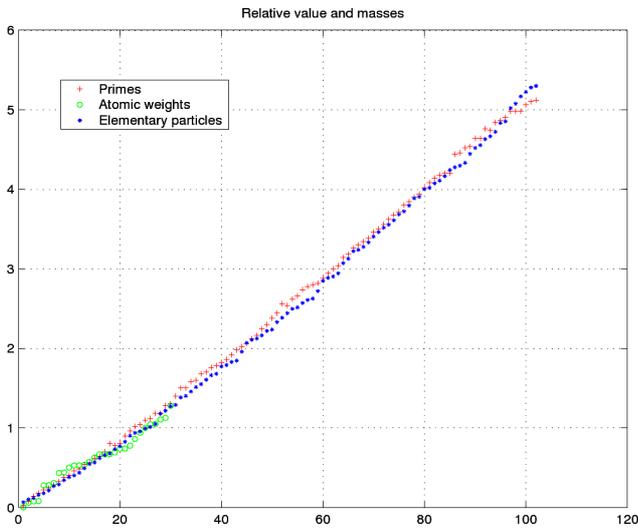


Fig. 7: Three normalized plots in ascending order of the relative atomic weight of 102 elements (+); 30 elementary particles (o); the first 102 prime numbers (*), starting with number 7. Each group is divided by entry number 25 of the group.

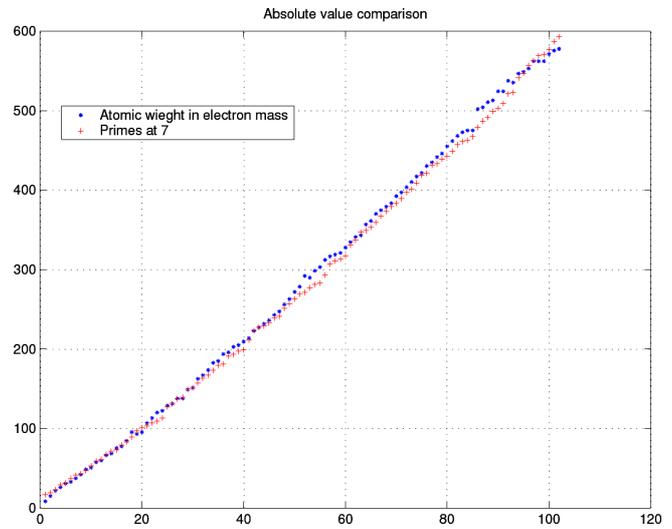


Fig. 8: Absolute-value comparison of the masses of chemical elements and primes. Primes starting from 7 (+) and relative masses of the chemical elements of the periodic table in units of Electron mass divided by (137×6) (*).

lowing the prime packing patterns as a result. We may also repeat the same for the masses of the *elementary particles* of physics which have hitherto defied many efforts to put a sense in the interpretation of their mass spectrum. To do this we shall arrange the various chemical elements of the periodic table (102 in total) and most of the elementary particles (30 in totals) in an ascending order of their masses (disregarding any other chemical property). We shall divide the masses of the chemical elements by the mass of the element say, number 25, in the list of ascending mass- which is Manganese (mass 55 protons) in order to get a relative value picture. The same is done with the group of elementary particles and these are divided by the mass of particle number 25 in the list namely the (Tau) particle (mass 1784 in MeV/c^2 units). Actual units do not matter here as we are only considering ratios. We then compare these with the list of primes arranged in ascending order too. Table 2 contains the data for the case of elementary particles. Masses of the chemical elements can be taken from any periodic table. The nearest prime figures in the table are

for information and not used in the plots. In Figure 8 an absolute value comparison for the elements is shown. The primes starts at 7 and the masses of the elements (in electron mass) are divided by 137×6 in order to get the two curves matching at the two ends.

For better fitting, the prime number series had to be started at number 7, not 1 as one might normally do. Comparison results are given in Figures 7 and 8. The trends are strikingly similar. The type of agreement must be a strong indication that the same packing rules are prevailing in all the cases.

3 Concluding remarks

We noticed that primes are closely connected to geometry and physics and this is dictated by the very properties of discrete space geometry like you can closely pack on a plane only seven balls to form a circle. This result and that of the cannon ball packing problem are found to be subsets of the dense packing problem. One clear link between primes and geom-

etry comes from the fact that all primes are generated by the formula $6n \pm 1$. When integers are associated with balls, this formula can be represented in the form of a long cylinder with primes lying along two opposite generator of the cylindrical surface.

Highly energetic particles bound together dynamically are more likely to have circular/spherical structures, and thus can follow the packing arrangements discussed in this article. It may be said now that the source of discreteness frequently observed in the energy levels of atoms and the correspondence between energy levels and prime numbers are only manifestations of this fact. The number of elements (balls) in each circular area or spherical leaf in the building up of a collection is proportional to n^2 . The energy of each would naturally be proportional n^2 too. Each constituent will thus carry $1/n^2$ of the energy and the jump of one constituent from one level to the other gives an energy change of $(1/n_1^2 - 1/n_2^2)$ as in the Ballmer series. The Bohr model for the atom relies on an integral number of wavelengths around a circumference, which in this case can be interpreted as integral number of balls, which makes the present model more realistic and easier to digest. The Bohr model was originally intended for the electrons, but later studies took this to concern the whole nucleus [8].

If the packing picture is carried down to the level of very elementary particles, we could speculate that the 2 and 3 circle solutions of the packing problem correspond to the 2 and 3 quarks constituent evidence found in experimental work and stated in the quark theory of elementary particles. Fast particles crossing the nucleus are normally used to probe the nucleus. The 6 pack with 3 balls along any diameter could very well be responsible for the conclusions of such measurements.

The plots of the mass growth (packing) of chemical elements and elementary particles (and hence all massive bodies), as shown here, follow very closely the rules of packing of spheres and also those of the prime numbers. Prime numbers or prime collections appear when it is not possible to divide a collection into symmetric (equal) parts and are hence more stable in structure. This makes the growth of primes to be naturally tied to the growth in the masses of condensed matter in its different phases. We also note that the prime character of a number is an independent property- more of an abstract physical property, and it is not a function of the base of the number system in use or the physical case that number might represent.

The eight fold rules frequently found in the behaviour pattern of chemical elements and elementary particles [4, 8] may now be suspected to be a consequence of the packing rules of similar spheres in space. We might even suggest that the successes of the Bohr Theory for the atom, the Ballmer series formula for energy levels and indeed the Schrödinger equation itself in predicting discrete behaviour in atoms and other entities, might be mainly due to the discretization of space implied in their formulations. In fact while Schrödinger equa-

tion has many solutions, those deemed correct have to obey the integrability condition which is essentially a discretization (normalization) of space condition. We mention also that in the solutions of Schrödinger equation, the main interest when finding a solution (the wave function) is the resulting number of discrete states along any radial or circumferential direction and not the actual form (function) of the solution. Not forgetting also that the most fruitful solutions of Schrödinger equation are those in circular no-Cartesian coordinates anyway.

4 Recommendations

More work is needed to reach more concrete, verifiable and useful results. Such work might investigate the origin of the various properties that distinguish groups of elementary particles like strangeness, charm etc in relation to the possible geometric shape/packing of their constituents. The circles and spheres in the present investigation are not referring to a static picture, but one formed by very fast moving particles that generated such shapes as a result of their own dynamic rules. Detailed position-energy calculations of various arrangements, as done on crystals for example, could be done here to pin point the reasons behind an elementary particle to become stable or unstable in the presence of external disturbances, and also the explanation of the various probabilities associated with different break-up scenarios of unstable particles.

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