From Inspired Guess to Physical Theory: Finding a Theory of Gravitation

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A theory of gravitation satisfying all experimental results was previously proposed in this journal. The dynamics was determined by a proposed Lagrangian. In this paper it is shown how this Lagrangian can be derived heuristically. A Newtonian approach is used, as well as other methods.

1 Introduction

A theory proposed in previous articles in this journal [1–4] relied on two postulates, one of which is that the dynamics of a system is determined by a Lagrangian,

$$L = -\,m_0 \left(c^2 + v^2\right) \exp \frac{R}{r}\,, \tag{1}$$

where m_0 is the *gravitational rest mass* of a test body moving at velocity $\mathbf{v}$ in the vicinity of a massive, central body of mass M, $\gamma = 1/\sqrt{1 - v^2/c^2}$ and $R = 2GM/c^2$ is the Schwarzschild radius of the central body.

This Lagrangian leads to equations of motion that satisfy all experimental observation of gravitational effects. It also leads to expressions for electromagnetic and nuclear interactions. In this regard it gives the fine spectrum of the hydrogen atom and the Yukawa potential for the nuclear force.

No explanation was given of how this Lagrangian had been determined, but only that its validity is confirmed by the consistency of its resultant equations of motion and agreement with experiment.

It is informative to show how such a Lagrangian can be derived. The procedure leads to an understanding of the creation and development of physical theories.

When a Lagrangian embodies the fundamentals of a physical model it cannot be derived from first principles. What is needed is an inspired guess to start with. The equations of motion derived from the initial Lagrangian are compared with observation. If they do not fit satisfactorily with the first try, then one adjusts the Lagrangian to conform closer to experimental results. This modelling cycle is repeated until a satisfactory agreement is found with observation.

In the case of the above Lagrangian various approaches are possible. We consider some of them.

2 Newton's approach

We follow a *Gedanken* speculation of how Isaac Newton would have derived a law of gravitation if he had been aware of the modern classical tests for a theory of gravitation.

The development of theories of gravitation at the beginning of the previous century is well documented [5, 6]. The essential test for a theory of gravitation at that time was whether it explained the anomalous perihelion precession of the orbit of Mercury, first calculated by Leverrier in 1859. This was satisfactorily explained by Einstein's theory of general relativity. Further predictions of this theory, i.e. the bending of light by a massive body and of gravitational redshift, have subsequently become part of the three benchmark tests for a model of gravitation.

2.1 Modern Newton

It is not generally known that Newton first derived his inverse square law of gravitation by first considering circular orbits [7, 8]. He applied Huygens's law for the acceleration in a circular orbit,

$$a = \frac{v^2}{r}\,, \tag{2}$$

and Kepler's third law to arrive at the inverse-square relation. He then proceeded to show in his *Philosophiae Naturalis Principia Mathematica* (there is some doubt about this [9]) that elliptical motion follows in general from this relation.

We follow a similar procedure by assuming a scenario along which Newton could have reasoned today to arrive at a refinement of his law of gravitation.

He would have been aware of the three classical tests for a theory of gravitation and that particles traveling near the speed of light obey relativistic mechanics. Following an iterative procedure he would have started with the simple model of circular orbits, derived the appropriate law of gravity, but modified to accommodate relativistic effects and then generalised it to include the other conical sections. It would finally be compared with other experimental results.

2.2 Finding a Lagrangian

For motion in a circular orbit under the gravitational attraction of a mass M one must have:

$$\frac{v^2}{r} = \frac{GM}{r^2}\,. \tag{3}$$

Because of relativistic considerations, the ratio v^2/c^2 must be compared relative to unity, i.e.

$$1 - \frac{v^2}{c^2} = 1 - \frac{GM}{r\,c^2}\,. \tag{4}$$

Note that (4) is not an approximation of (2) for $v \ll c$. If we surmise that the inverse square law is only valid for $r \gg R$, one could incorporate higher order gravitational effects by generalising the right-hand side of (4) to a polynomial. Furthermore, to allow other motion besides circles, we multiply the right-hand side by an arbitrary constant K:

$$\begin{aligned} 1-\frac{v^2}{c^2} &= \left(1+\frac{a'R}{r}+\frac{b'R^2}{r^2}+\ldots\right)K\,, \\ &= KP'(r)\,, \end{aligned} \tag{5}$$

or

$$\left(1-\frac{v^2}{c^2}\right)P(r) = K\,, \tag{6}$$

where

$$P(r) = 1+\frac{aR}{r}+\frac{bR^2}{r^2}\ldots, \tag{7}$$

is the inverse of $P'(r)$.

In order to compare (6) with experiment, we have to convert it to some standard form in physics. To do this we first rewrite (6) as:

$$(1-K)\,\frac{c^2}{2} = \frac{v^2}{2}-\frac{GMa}{r}-\frac{av^2R}{2r}+\ldots \tag{8}$$

If we multiply this equation by a constant, m_0, with the dimension of mass, we obtain a conservation equation with the dimensions of energy:

$$(1-K)\,\frac{m_0\,c^2}{2} = \frac{m_0\,v^2}{2}-\frac{GMm_0\,a}{r}-\frac{m_0\,a\,v^2R}{2r}+\ldots \tag{9}$$

For $r \gg R$, this equation must approach the Newtonian limit:

$$\frac{m_0\,v^2}{2}-\frac{m_0MGa}{r} = E_N\,, \tag{10}$$

where E_N is the total Newtonian energy. Comparison of (10) with the Newtonian expression gives $a = 1$.

To simplify the notation, we define a constant E with dimensions of energy, such that

$$K = \frac{E}{m_0c^2}\,. \tag{11}$$

From (6),

$$E = m_0c^2\left(1-\frac{v^2}{c^2}\right)P(r)\,. \tag{12}$$

If we consider (12) as the total energy of the system, we can find a corresponding Lagrangian by separating the potential and kinetic energies:

$$\begin{aligned} T &= -\,m_0\,v^2P(r)\,, \\ V &= m_0\,c^2P(r)\,. \end{aligned}$$

The corresponding Lagrangian is therefore:

$$L = T-V = -\,m_0\left(c^2+v^2\right)\,P(r)\,. \tag{13}$$

Applying the Euler-Lagrange equations to this Lagrangian one can find the equations of motion of the system. The conservation of energy (12) follows again, while for the conservation of angular momentum we find

$$P(r)\,r^2\,\dot\theta \;=\; \text{constant} \;=\, h. \tag{14}$$

The equations of motion for the system can then be derived from (12) and (14) as a generalised Kepler problem. From these equations one finds a differential equation of motion of the form

$$\frac{d\theta}{du} = Au^2+Bu+C\,, \tag{15}$$

where

$$u \;=\; \frac{1}{r}\,,$$

$$A \;=\; bR^2\,\frac{4-E}{2h}-1\,, \tag{16}$$

$$B \;=\; \frac{R\,(2-E)}{h^2}\,, \tag{17}$$

$$C \;=\; \frac{1-E}{h^2}\,. \tag{18}$$

The convention $m_0 = c = 1$ was used, and terms higher than R^2/r^2 were ignored.

2.3 Perihelion precession

In the case of an ellipse, the presence of the coefficient A gives rise to a precession of the perihelion. For one revolution this can be calculated as:

$$\frac{6b\pi cR}{\bar a(1-e^2)}\,, \tag{19}$$

where $\bar a$ is the semi-major axis and e is the eccentricity of the ellipse. Comparison with the observed value for Mercury gives $b = 1/2$. With this result the polynomial of (7) becomes:

$$P(r) = 1+\frac{R}{r}+\frac{R^2}{2r^2}+\ldots \tag{20}$$

Equation (20) could be regarded as simply a fit to experimental data. The theoretical physicist, however, will look for a pattern or a generalisation of some underlying physical law. The form of the equation leads one to propose that the above terms are the first three terms in the Taylor expansion of

$$P(r) = \exp\frac{R}{r}\,. \tag{21}$$

Confirmation of this form, which is aesthetically more acceptable, must come from other experimental results, such as the bending of light by a massive object. This is shown in the first article referred to above [1].

The Lagrangian of (13) can now be rewritten in the form of (1):

$$L = -\,m_0\left(c^2+v^2\right)\exp\frac{R}{r}\,, \tag{22}$$

or in terms of the potential Φ as

$$L = -\,m_0\left(c^2 + v^2\right)\exp\frac{2\Phi}{c^2}\,. \tag{23}$$

The conservation of energy equation (12) can be written as

$$E = m_0\,c^2\,\frac{e^{R/r}}{\gamma^2}\,. \tag{24}$$

We define a variable *gravitational mass* as

$$m = \frac{m_0}{\gamma^2}, \tag{25}$$

so that (24) can also be written as

$$E = m\,c^2\,e^{R/r}. \tag{26}$$

3 A gravitational redshift approach

We continue with the hypothetical Newton, but starting from another experimental observation. In the presence of a body of mass M a photon undergoes a frequency shift relative to its frequency ν_0 in the absence of the body:

$$\nu = \nu_0\left(1 - \frac{R}{2r}\right),$$

where ν_0 is an invariant.

In line with our inspired guess approach, we surmise that the right-hand side of this equation is a first order approximation to

$$\nu = \nu_0\,e^{-R/2r}, \tag{27}$$

or

$$\nu_0 = \nu\,e^{R/2r}. \tag{28}$$

Substituting time for the frequency, $\nu = 1/t$ and rearranging:

$$dt = B\,e^{R/2r}d\tau\,, \tag{29}$$

where $d\tau$ is an invariant time interval, or proper time, and B is a proportionality constant. Substituting the special relativity relation $dt = \gamma d\tau$ in (29),

$$\frac{1}{B} = \frac{e^{R/2r}}{\gamma}\,. \tag{30}$$

This is a conservation equation involving the variables r, v and M. In order to relate this equation to the classical conservation of energy equation and its Newtonian limit, the equation must be squared and multiplied by $m_0\,c^2$:

$$\frac{m_0\,c^2}{B^2} = m_0\,c^2\,\frac{e^{R/r}}{\gamma^2}\,. \tag{31}$$

This is the same equation as (24) for $E = m_0c^2/B^2$.

From (11) we note that $B^2 = 1/K$. Separating the kinetic and potential energy terms we again find the Lagrangian of (1).

4 An Einstein approach

It is understandable that the large corpus of publications on general relativity (GR) over the past few decades tend to underrate the heuristic approach, or inspired guesses, which are used to derive the field equations of GR. The classic texts do not. On page 152 of Weinberg's *Gravitation and Cosmology* [10] the author emphasises the guesswork that leads to the field equations. Eddington [11, p.82] mentions that "This preliminary argument need not be rigorous; the final test is whether the formulae suggested by it satisfy the equations to be solved". This is a classical heuristic argument.

One can therefore wonder why the heuristic derivation was not continued to generalise the metric of GR,

$$ds^2 = \left(1 - \frac{R}{r}\right)dt^2 - \\ - \frac{1}{1 - \frac{R}{r}}\,dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\,d\phi^2, \tag{32}$$

to an exponential form:

$$ds^2 = e^{-R/r}dt^2 - e^{R/r}(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2)\,. \tag{33}$$

The equations of motion derived from this metric are the same as those derived from the Lagrangian of (1), but are conceptually and mathematically simpler [1]. From the resulting conservation equations one can, similarly to the procedures above, derive our Lagrangian.

5 Nordström's first theory

Although not an example of a heuristic derivation, Gunnar Nordström's first theory [12, 13] is an intriguing example of how theories of gravitation could have taken a different direction in 1912.

Nordström's theory, a noncovariant one, is based on a Lagrangian,

$$L = \exp\frac{R}{2r}\,. \tag{34}$$

In the case of a static, spherically symmetrical field the Lagrangian gives a conservation equation,

$$\gamma\exp\left(-\frac{R}{2r}\right) = A_N\,. \tag{35}$$

Comparison with (30) shows that $A_N = B$. Nordström's first theory therefore gives the same conservation of energy equation as our theory.

The absence of the $\left(c^2 + v^2\right)$ term in Nordström's Lagrangian accounts for its difference from our theory and Nordström's wrong predictions. This shows up in his conservation of angular momentum,

$$r^2\,\frac{d\theta}{dt} = h\,, \tag{36}$$

where h = constant.

Nordström's theory [14] also gives a variation of mass,

$$m = m_0\, e^{-R/2r}. \tag{37}$$

From (11) and (26) our theory gives

$$m = K m_0\, e^{-R/r}. \tag{38}$$

The close correlation between our theory and that of Nordström raises the possibility of Nordström, or anyone else reading his paper of 1912, deriving the Lagrangian of (1). If this had happened, and the resultant agreement with Mercury's perihelion precession were found, then the study of gravitation could have followed a different direction.

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