Van Aubel's Theorem in the Einstein Relativistic Velocity Model of Hyperbolic Geometry

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In this note, we present a proof to the Van Aubel Theorem in the Einstein Relativistic Velocity Model of Hyperbolic Geometry.

1 Introduction

Hyperbolic Geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. There are known many models for Hyperbolic Geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. Here, in this study, we give hyperbolic version of Van Aubel theorem. The well-known Van Aubel theorem states that if ABC is a triangle and AD, BE, CF are concurrent cevians at P, then $\frac{AP}{PD} = \frac{AE}{EC} + \frac{AF}{FB}$ (see [1, p. 271]).

Let D denote the complex unit disc in complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta}\frac{z_0 + z}{1 + \overline{z_0}z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyrotranslation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0}z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyrocommutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyrovector space $(G, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:

1. $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.
2. G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

   (G1) $1 \otimes \mathbf{a} = \mathbf{a}$

   (G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

   (G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

   (G4) $\dfrac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \dfrac{\mathbf{a}}{\|\mathbf{a}\|}$

   (G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$

   (G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$

3. Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of onedimensional "vectors"

   $$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

   with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

   (G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$

   (G8) $\|\mathbf{a} \oplus \mathbf{b}\| \le \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Definition 1. *Let ABC be a gyrotriangle with sides a, b, c in an Einstein gyrovector space $(V_s, \oplus, \otimes)$, and let h_a, h_b, h_c be three altitudes of ABC drawn from vertices A, B, C perpendicular to their opposite sides a, b, c or their extension, respectively. The number*

$$S_{ABC} = \gamma_a a \gamma_{h_a} h_a = \gamma_b b \gamma_{h_b} h_b = \gamma_c c \gamma_{h_c} h_c$$

is called the gyrotriangle constant of gyrotriangle ABC (here $\gamma_{\mathbf{v}} = \dfrac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{s^2}}}$ is the gamma factor).

(See [2, p. 558].)

Theorem 1. ***(The Gyrotriangle Constant Principle)***
Let A_1BC and A_2BC be two gyrotriangles in a Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$, $A_1 \neq A_2$ such that the two gyrosegments A_1A_2 and BC, or their extensions, intersect at a point $P \in \mathbb{R}^2_s$. Then,

$$\frac{\gamma_{|A_1P|}\,|A_1P|}{\gamma_{|A_2P|}\,|A_2P|} = \frac{S_{A_1BC}}{S_{A_2BC}}.$$

(See [2, p. 563].)

Theorem 2. ***(The Hyperbolic Theorem of Menelaus in Einstein Gyrovector Space)***
Let $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ be three non-gyrocollinear points in an Einstein gyrovector space $(V_s, \oplus, \otimes)$. If a gyroline meets the sides of gyrotriangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ at points $\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{23}$, then

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\,\|\ominus\mathbf{a}_1 \oplus \mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\,\|\ominus\mathbf{a}_2 \oplus \mathbf{a}_{12}\|} \cdot \frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}}\,\|\ominus\mathbf{a}_2 \oplus \mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}}\,\|\ominus\mathbf{a}_3 \oplus \mathbf{a}_{23}\|} \cdot$$

$$\frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}}\,\|\ominus\mathbf{a}_3 \oplus \mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}}\,\|\ominus\mathbf{a}_1 \oplus \mathbf{a}_{13}\|} = 1$$

(See [2, p. 463].)

Theorem 3. ***(The Gyrotriangle Bisector Theorem)***
Let ABC be a gyrotriangle in an Einstein gyrovector space $(V_s, \oplus, \otimes)$, and let P be a point lying on side BC of the gyrotriangle such that AP is a bisector of gyroangle $\measuredangle BAC$. Then,

$$\frac{\gamma_{|BP|}\,|BP|}{\gamma_{|PC|}\,|PC|} = \frac{\gamma_{|AB|}\,|AB|}{\gamma_{|AC|}\,|AC|}$$

(See [3, p. 150].) For further details we refer to the recent book of A. Ungar [2].

2 Main results

In this section, we prove Van Aubel's theorem in hyperbolic geometry.

Theorem 4. *If the point P does lie on any side of the hyperbolic triangle ABC, and BC meets AP in D, CA meets BP in E, and AB meets CP in F, then*

$$\frac{\gamma_{|AP|}\,|AP|}{\gamma_{|PD|}\,|PD|} = \frac{\gamma_{|BC|}\,|BC|}{2} \cdot \frac{\gamma_{|AE|}\,|AE|}{\gamma_{|EC|}\,|EC|} \cdot \frac{1}{\gamma_{|BD|}\,|BD|} +$$

$$\frac{\gamma_{|BC|}\,|BC|}{2} \cdot \frac{\gamma_{|FA|}\,|FA|}{\gamma_{|FB|}\,|FB|} \cdot \frac{1}{\gamma_{|CD|}\,|CD|}.$$

Proof. If we use the Menelaus's theorem in the h-triangles ADC and ABD for the h-lines BPE, and CPF respectively, then

$$\frac{\gamma_{|AP|}\,|AP|}{\gamma_{|PD|}\,|PD|} = \frac{\gamma_{|AE|}\,|AE|}{\gamma_{|EC|}\,|EC|} \cdot \frac{\gamma_{|BC|}\,|BC|}{\gamma_{|BD|}\,|BD|} \tag{1}$$

and

$$\frac{\gamma_{|AP|}\,|AP|}{\gamma_{|PD|}\,|PD|} = \frac{\gamma_{|FB|}\,|FB|}{\gamma_{|FA|}\,|FA|} \cdot \frac{\gamma_{|BC|}\,|BC|}{\gamma_{|CD|}\,|CD|} \tag{2}$$

From (1) and (2), we have

$$2 \cdot \frac{\gamma_{|AP|}\,|AP|}{\gamma_{|PD|}\,|PD|} = \frac{\gamma_{|AE|}\,|AE|}{\gamma_{|EC|}\,|EC|} \cdot \frac{\gamma_{|BC|}\,|BC|}{\gamma_{|BD|}\,|BD|} +$$

$$\frac{\gamma_{|FA|}\,|FA|}{\gamma_{|FB|}\,|FB|} \cdot \frac{\gamma_{|BC|}\,|BC|}{\gamma_{|CD|}\,|CD|},$$

the conclusion follows. □

Corollary 1. *Let G be the centroid of the hyperbolic triangle ABC, and D, E, F are the midpoints of hyperbolic sides BC, CA, and AC respectively. Then,*

$$\frac{\gamma_{|AG|}\,|AG|}{\gamma_{|GD|}\,|GD|} = \frac{\gamma_{|BC|}\,|BC|}{2}\left[\frac{1}{\gamma_{|BD|}\,|BD|} + \frac{1}{\gamma_{|CD|}\,|CD|}\right]. \tag{3}$$

Proof. If we use theorem 4 for the triangle ABC and the centroid G, we have

$$\frac{\gamma_{|AG|}\,|AG|}{\gamma_{|GD|}\,|GD|} = \frac{\gamma_{|BC|}\,|BC|}{2} \cdot \frac{\gamma_{|AE|}\,|AE|}{\gamma_{|EC|}\,|EC|} \cdot \frac{1}{\gamma_{|BD|}\,|BD|} +$$

$$\frac{\gamma_{|BC|}\,|BC|}{2} \cdot \frac{\gamma_{|FA|}\,|FA|}{\gamma_{|FB|}\,|FB|} \cdot \frac{1}{\gamma_{|CD|}\,|CD|},$$

the conclusion follows. □

Corollary 2. *Let I be the incenter of the hyperbolic triangle ABC, and let the angle bisectors be AD, BE, and CF. Then,*

$$\frac{\gamma_{|AI|}\,|AI|}{\gamma_{|ID|}\,|ID|} = \frac{1}{2}\left[\frac{\gamma_{|AB|}\,|AB|}{\gamma_{|BD|}\,|BD|} + \frac{\gamma_{|AC|}\,|AC|}{\gamma_{|CD|}\,|CD|}\right]. \tag{4}$$

Proof. If we use theorem 3 for the triangle ABC, we have

$$\frac{\gamma_{|AE|}\,|AE|}{\gamma_{|EC|}\,|EC|} = \frac{\gamma_{|AB|}\,|AB|}{\gamma_{|BC|}\,|BC|}, \text{ and}$$

$$\frac{\gamma_{|AF|}\,|AF|}{\gamma_{|FB|}\,|FB|} = \frac{\gamma_{|AC|}\,|AC|}{\gamma_{|BC|}\,|BC|}. \tag{5}$$

If we use theorem 4 for the triangle ABC and the incenter I, we have

$$\begin{aligned}\frac{\gamma_{|AI|}\,|AI|}{\gamma_{|ID|}\,|ID|} = &\frac{\gamma_{|BC|}\,|BC|}{2} \cdot \frac{\gamma_{|AE|}\,|AE|}{\gamma_{|EC|}\,|EC|} \cdot \frac{1}{\gamma_{|BD|}\,|BD|} + \\ &\frac{\gamma_{|BC|}\,|BC|}{2} \cdot \frac{\gamma_{|FA|}\,|FA|}{\gamma_{|FB|}\,|FB|} \cdot \frac{1}{\gamma_{|CD|}\,|CD|}.\end{aligned} \tag{6}$$

From (5) and (6), we have

$$\frac{\gamma_{|AI|}\,|AI|}{\gamma_{|ID|}\,|ID|} = \frac{\gamma_{|BC|}\,|BC|}{2} \cdot \frac{\gamma_{|AB|}\,|AB|}{\gamma_{|BC|}\,|BC|} \cdot \frac{1}{\gamma_{|BD|}\,|BD|} +$$

$$\frac{\gamma_{|BC|}\,|BC|}{2} \cdot \frac{\gamma_{|AC|}\,|AC|}{\gamma_{|BC|}\,|BC|} \cdot \frac{1}{\gamma_{|CD|}\,|CD|},$$

the conclusion follows. □

The Einstein relativistic velocity model is another model of hyperbolic geometry. Many of the theorems of Euclidean geometry are relatively similar form in the Einstein relativistic velocity model, Aubel's theorem for gyrotriangle is an example in this respect. In the Euclidean limit of large s, $s \to \infty$, gamma factor γ_v reduces to 1, so that the gyroequality (1) reduces to the

$$\frac{|AP|}{|PD|} = \frac{|BC|}{2}\left[\frac{|AE|}{|EC|}\cdot\frac{1}{|BD|} + \frac{|FA|}{|FB|}\cdot\frac{1}{|CD|}\right]$$

in Euclidean geometry. We observe that the previous equality is a equivalent form to the Van Aubel's theorem of euclidian geometry.

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