Fine Structure Constant as a Mirror of Sphere Geometry

Janez Špringer
Cankarjeva cesta 2, 9250 Gornja Radgona, Slovenia, EU. E-mail: janez_springer@t-2.net

A path is defined as the vector's sum of the translation and rotation component of the length unit belonging to the mass entity in motion on the sphere. The fine structure constant is an irrational number being a mirror of the path complexity as well as the sphere curvature where the path is made. The inverse value in the Euclidean plane yields $\alpha^{-1} = \sqrt{\pi^2 + 137^2}$. The inverse fine structure constant on the elliptic sphere is smaller and on the hyperbolic sphere is greater. The electron in the Hydrogen atom should move on the elliptic sphere of the radius of 3679 Compton wavelengths of the electron according to the CODATA 2012 recommended empirical value $\alpha^{-1} = 137.035999074$. Such a small sphere radius implies the heterogeneous curvature of the present universe.

1 Theoretical background

In motion is an entity having some mass. Respecting Compton the length unit is attributed to that mass:

$$\lambda = \frac{h}{mc} = 1\,. \tag{1}$$

The infinite mass and zero length unit are objectively unreachable. Nevertheless both can be theoretically approached arbitrarily close by the sufficiently great finite mass.

A curved motion obeys the path complexity: it has the translation and rotation component. Describing the curved path the length unit becomes not only the translation unit but the rotation unit, too. By the circumference of a circle concluded path s, for instance, only apparently equals the translation n, actually it is greater for the average rotation π made around the start point of the length unit:

$$\pi = \frac{0 + 2\pi \times 1}{2}\,. \tag{2}$$

The actual path is the vectorial sum of both components: the rotation π as well as translation n:

$$\vec{s} = \vec{\pi} + \vec{n}\,. \tag{3}$$

The total rotation of the length unit π equals the total Berry phase at spin $\frac{1}{2}$ [1].

1.1 Path in the Euclidean plane

By the circumference of a circle concluded path s in the Euclidean plane is calculated with the help of Pythagoras' theorem:

$$s^2 = \pi^2 + n^2. \tag{4}$$

1.2 Path on the elliptic sphere

By the circumference of a circle concluded path s on the elliptic sphere is calculated with the help of the spherical law of cosines.

On the elliptic sphere of radius R holds:

$$\cos\frac{s}{R} = \cos\frac{\pi}{R}\,\cos\frac{n}{R}\,, \tag{5}$$

$$\cos x = \sqrt{1-\sin^2 x}\,, \tag{6}$$

$$\frac{1}{R^2} = \frac{1}{k_1^2\pi^2} + \frac{1}{k_2^2 n^2} - \frac{k_3^2 s^2}{k_1^2\pi^2 \times k_2^2 n^2} = \frac{k_1^2\pi^2 + k_2^2 n^2 - k_3^2 s^2}{k_1^2\pi^2 \times k_2^2 n^2}\,. \tag{7}$$

The coefficients are expressed as

$$k_1 = \frac{\sin\frac{\pi}{R}}{\frac{\pi}{R}}\,, \quad k_2 = \frac{\sin\frac{n}{R}}{\frac{n}{R}} \quad \text{and} \quad k_3 = \frac{\sin\frac{s}{R}}{\frac{s}{R}}\,. \tag{8}$$

They are arranged by size

$$1 \geqslant k_1 \geqslant k_2 \geqslant k_3\,. \tag{9}$$

In the case of R^2 being a positive number Pythagoras' theorem holds only exceptionally. The next condition has to be satisfied:

$$k_1^2\pi^2 + k_2^2 n^2 \geqslant k_3^2 s^2 \quad \text{or} \quad \frac{k_1^2}{k_3^2}\pi^2 + \frac{k_2^2}{k_3^2} n^2 \geqslant s^2. \tag{10}$$

The ratios of coefficients $\frac{k_1^2}{k_3^2}$ and $\frac{k_2^2}{k_3^2}$ are according to (non) equation (9) greater than 1 or at least equal 1, therefore we write:

$$\frac{k_1^2}{k_3^2}\pi^2 + \frac{k_2^2}{k_3^2} n^2 \geqslant \pi^2 + n^2 \geqslant s^2. \tag{11}$$

At the finite elliptic sphere radius R Pythagoras' theorem fails, because at non-equal coefficients (9) the square area upon hypotenuse is smaller than the sum of square areas upon catheters:

$$s^2 < \pi^2 + n^2. \tag{12}$$

At $R = \infty$ and equal coefficients (9) the elliptic sphere transforms into the Euclidean plane and Pythagoras' theorem begins to rule again (4).

1.2.1 Approximation for cos x

Hardy's approximation [2] is close to the function $\cos\frac{t}{r}$:

$$H\left(\frac{2t}{\pi R}\right) = \cos\frac{t}{R} \approx 1 - \frac{\left(\frac{2t}{\pi R}\right)^2}{\frac{2t}{\pi R} + \left(1 - \frac{2t}{\pi R}\right)\sqrt{\frac{2-\frac{2t}{\pi R}}{3}}}. \tag{13}$$

At very large R Hardy's approximation can be simplified:

$$H\left(\frac{2t}{\pi R}\right) = \cos\frac{t}{R} \approx 1 - \left(\frac{2t}{\pi R}\right)^2. \quad (14)$$

The spherical law of cosines (5) with the help of the simplified Hardy approximation (14) enables to calculate the approximate value of the sphere radius in cases of a tiny curvature where Pythagoras' theorem approximately rules. The explicit relation is expressed as

$$R^2 \approx \frac{(2n)^2}{n^2 + \pi^2 - s^2}. \quad (15)$$

The similar approximation is obtained with the help of equation (7) at the assumption of coefficients approximate equality:

$$1 \approx k_1 \approx k_2 \approx k_3. \quad (16)$$

Then the sphere radius is expressed as

$$R^2 \approx \frac{(\pi n)^2}{n^2 + \pi^2 - s^2}. \quad (17)$$

1.3 Path on the hyperbolic sphere

By the circumference of a circle concluded path s on the hyperbolic sphere is calculated with the help of the hyperbolic law of cosines.

On the hyperbolic sphere of radius R holds:

$$\cosh\frac{s}{R} = \cosh\frac{\pi}{R}\cosh\frac{n}{R}, \quad (18)$$

$$\cosh x = \sqrt{1 + \sinh^2 x}, \quad (19)$$

$$\frac{1}{R^2} = -\frac{1}{k_1^2\pi^2} - \frac{1}{k_2^2 n^2} + \frac{k_3^2 s^2}{k_1^2\pi^2 \times k_2^2 n^2} = \frac{-k_1^2\pi^2 - k_2^2 n^2 + k_3^2 s^2}{k_1^2\pi^2 \times k_2^2 n^2}. \quad (20)$$

The coefficients are expressed as

$$k_1 = \frac{\sinh\frac{\pi}{R}}{\frac{\pi}{R}}, \quad k_2 = \frac{\sinh\frac{n}{R}}{\frac{n}{R}} \quad \text{and} \quad k_3 = \frac{\sinh\frac{s}{R}}{\frac{s}{R}}. \quad (21)$$

They are arranged by size

$$1 \leqslant k_1 \leqslant k_2 \leqslant k_3. \quad (22)$$

In the case of R^2 being a positive number Pythagoras' theorem holds only exceptionally.

The next condition has to be satisfied:

$$k_3^2 s^2 \geqslant k_1^2\pi^2 + k_2^2 n^2 \quad or \quad s^2 \geqslant \frac{k_1^2}{k_3^2}\pi^2 + \frac{k_2^2}{k_3^2}n^2. \quad (23)$$

The ratios of coefficients k_1^2/k_3^2 and k_2^2/k_3^2 are according to (non)equation (22) smaller than 1 or at most equal 1, therefore we write:

$$\frac{k_1^2}{k_3^2}\pi^2 + \frac{k_2^2}{k_3^2}n^2 \leqslant \pi^2 + n^2 \leqslant s^2. \quad (24)$$

At the finite hyperbolic sphere radius R Pythagoras' theorem fails, because at non-equal coefficients (22) the square area upon hypotenuse is greater than the sum of square areas upon catheters:

$$s^2 > \pi^2 + n^2. \quad (25)$$

At $R = \infty$ and equal coefficients (22) the hyperbolic sphere transforms into the Euclidean plane and Pythagoras' theorem begins to rule again (4).

2 Fine structure constant and sphere radius

In the ground state of the Hydrogen atom the electron path around the nucleus equals the ratio of the Compton wavelength of the electron λ and the fine structure constant α. The wavelength equals the unit, so the circular path equals the inverse fine structure constant:

$$s = \alpha^{-1}. \quad (26)$$

2.1 Inverse fine structure constant on the non-Euclidean sphere and Euclidean plane

At the finite sphere radius R two possibilities are allowed according the non-equations (12) and (25).

On the elliptic sphere holds:

$$\alpha^{-2} < \pi^2 + n^2. \quad (27)$$

On the hyperbolic sphere holds:

$$\alpha^{-2} > \pi^2 + n^2. \quad (28)$$

At $R = \infty$ both non-Euclidean spheres transform into the Euclidean plane and according to the equation (4) holds:

$$\alpha^{-2} = \pi^2 + n^2. \quad (29)$$

2.2 Calculation of the theoretical inverse fine structure constant in the Euclidean plane

In the hydrogen atom the number $n = 137$ is to the inverse fine structure constant α^{-1} the closest natural number which concludes the start and end point of Bohr orbit. The number π is the total average rotation component of the length unit.

The theoretical inverse fine structure constant in the Euclidean plane is calculated with the help of the equation (29). Its value is an irrational number:

$$\alpha^{-1}{}_{\text{EUCLID}} = \sqrt{n^2 + \pi^2} \approx 137.036015720. \quad (30)$$

2.3 Calculation of the sphere radius on the atomic level

The inverse fine structure constant should be according to the equations (27) and (28) on the elliptic sphere smaller and on the hyperbolic sphere greater than $\alpha^{-1}_{\text{EUCLID}}$.

The recommended CODATA 2012 value of the inverse fine structure constant is smaller than the theoretical value in the Euclidean plane:

$$\alpha^{-1}_{\text{CODATA}} = 137.035999074 < \alpha^{-1}_{\text{EUCLID}} \approx 137.036015720. \quad (31)$$

This implies the elliptic sphere in the Hydrogen atom.

The calculus of the radius of the elliptic sphere with the help of the equation (5) yields:

$$R = 3679 \text{ Compton wavelengths of the electron.} \quad (32)$$

The estimate of the radius of the elliptic sphere with the help of the simplified Hardy approximation (15) yields a little bit greater value:

$$R \approx \frac{2.137}{\sqrt{137^2 + \pi^2 - \alpha^{-2}{}_{\text{CODATA}}}} = 4057. \quad (33)$$

2.4 Estimation of the inverse fine structure constant on the macro level

Let us consider the radius of the observable universe of about 4×10^{26}m [3] as the sphere radius:

$$R \approx 2 \times 10^{38} \text{ Compton wavelengths of the electron.} \quad (34)$$

This is a huge radius. A common calculator supports the spherical law of cosines only for radius up to $\sim 10^{15}$ Compton wavelengths of the electron.

Fortunately a huge sphere radius is given by the simplified Hardy approximation (15) in the explicit relation with the inverse fine structure constant:

$$R^2 \approx \frac{(2.137)^2}{\pi^2 + 137^2 - \alpha^{-2}}, \quad (35)$$

$$\alpha^{-1} \approx \sqrt{\pi^2 + 137^2 \left(1 - \frac{4}{R^2}\right)} = \sqrt{\pi^2 + 137^2 \left(1 - 10^{-76}\right)} \approx$$
$$\approx \sqrt{\pi^2 + 137^2}. \quad (36)$$

If the sphere curvature on the atomic level equals the curvature of the hypothetical elliptic observable universe, the inverse fine structure constant should not significantly differ from the theoretical constant in the Euclidean plane.

3 Conclusion

If the inverse fine structure constant is a mirror of the path complexity as well as the curvature of the sphere where the path is made, its theoretical inverse value in the Euclidean plane $\alpha^{-1} = \sqrt{\pi^2 + 137^2}$ and the recommended empirical CODATA 2012 value $\alpha^{-1} = 137.035999074$ express the electron motion on the elliptic sphere of the radius of 3679 Compton wavelengths of the electron. This implies a huge curvature of the atomic world. If the sphere curvatures in the atomic and the macro-world would be the same, the inverse fine structure constant should not significantly differ from the theoretical one in the Euclidean plane.

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