The Elastodynamics of the Spacetime Continuum as a Framework for Strained Spacetime

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We derive the elastodynamics of the spacetime continuum by applying continuum mechanical results to strained spacetime. Based on this model, a stress-strain relation is derived for the spacetime continuum. From the kinematic relations and the equilibrium dynamic equation of the spacetime continuum, we derive a series of wave equations: the displacement, dilatational, rotational and strain wave equations. Hence energy propagates in the spacetime continuum as wave-like deformations which can be decomposed into dilatations and distortions. Dilatations involve an invariant change in volume of the spacetime continuum which is the source of the associated rest-mass energy density of the deformation, while distortions correspond to a change of shape of the spacetime continuum without a change in volume and are thus massless. The deformations propagate in the continuum by longitudinal and transverse wave displacements. This is somewhat reminiscent of wave-particle duality, with the transverse mode corresponding to the wave aspects and the longitudinal mode corresponding to the particle aspects. A continuity equation for deformations of the spacetime continuum is derived, where the gradient of the massive volume dilatation acts as a source term. The nature of the spacetime continuum volume force and the inhomogeneous wave equations need further investigation.

1 Introduction

Strained spacetime has been explored recently by Millette [1] from a continuum mechanical and general relativistic perspective, and by Tartaglia *et al* in the cosmological context, as an extension of the spacetime Lagrangian, to obtain a generalized Einstein equation [2, 3].

As shown in [1], the applied stresses from the energy-momentum stress tensor result in strains in the spacetime continuum. The presence of strains as a result of applied stresses is an expected continuum mechanical result. The strains result in a deformation of the continuum which can be modeled as a change in the underlying geometry of the continuum. The geometry of the spacetime continuum of General Relativity resulting from the energy-momentum stress tensor can thus be seen as a representation of the deformation of the spacetime continuum resulting from the strains generated by the energy-momentum stress tensor.

In this paper, we examine in greater details the elastodynamics of the spacetime continuum as a framework for describing strained spacetime.

2 Elastodynamics of the Spacetime Continuum

2.1 Model of the Elastodynamics of the Spacetime Continuum

The spacetime continuum (*STC*) is modelled as a four-dimensional differentiable manifold endowed with a metric $g_{\mu\nu}$. It is a continuum that can undergo deformations and support the propagation of such deformations. A continuum that is deformed is strained.

An infinitesimal element of the unstrained continuum is characterized by a four-vector x^μ, where $\mu = 0, 1, 2, 3$. The time coordinate is $x^0 \equiv ct$.

A *deformation* of the spacetime continuum corresponds to a state of the *STC* in which its infinitesimal elements are displaced from their unstrained position. Under deformation, the infinitesimal element x^μ is displaced to a new position $x^\mu + u^\mu$, where u^μ is the displacement of the infinitesimal element from its unstrained position x^μ.

The spacetime continuum is approximated by a deformable linear elastic medium that obeys Hooke's law. For a general anisotropic continuum in four dimensions [4, see pp. 50–53],

$$E^{\mu\nu\alpha\beta}\varepsilon_{\alpha\beta} = T^{\mu\nu} \tag{1}$$

where $\varepsilon_{\alpha\beta}$ is the strain tensor, $T^{\mu\nu}$ is the energy-momentum stress tensor, and $E^{\mu\nu\alpha\beta}$ is the elastic moduli tensor.

The spacetime continuum is further assumed to be isotropic and homogeneous. This assumption is in agreement with the conservation laws of energy-momentum and angular momentum as expressed by Noether's theorem [5, see pp. 23–30]. For an isotropic medium, the elastic moduli tensor simplifies to [4]:

$$E^{\mu\nu\alpha\beta} = \lambda_0(g^{\mu\nu}g^{\alpha\beta}) + \mu_0(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}) \tag{2}$$

where λ_0 and μ_0 are the Lamé elastic constants of the spacetime continuum. μ_0 is the shear modulus (the resistance of the continuum to *distortions*) and λ_0 is expressed in terms of κ_0, the bulk modulus (the resistance of the continuum to *dilatations*) according to

$$\lambda_0 = \kappa_0 - \mu_0/2 \tag{3}$$

in a four-dimensional continuum. A *dilatation* corresponds to a change of volume of the spacetime continuum without a change of shape while a *distortion* corresponds to a change of shape of the spacetime continuum without a change in volume.

2.2 Stress-Strain Relation of the Spacetime Continuum

Substituting Eq.(2) into Eq.(1), we obtain the stress-strain relation for an isotropic and homogeneous spacetime continuum

$$2\mu_0 \varepsilon^{\mu\nu} + \lambda_0 g^{\mu\nu}\varepsilon = T^{\mu\nu} \tag{4}$$

where

$$\varepsilon = \varepsilon^{\alpha}{}_{\alpha} \tag{5}$$

is the trace of the strain tensor obtained by contraction. The volume dilatation ε is defined as the change in volume per original volume [6, see pp. 149–152] and is an invariant of the strain tensor.

It is interesting to note that the structure of Eq.(4) is similar to that of the field equations of General Relativity, viz.

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -K T^{\mu\nu} \tag{6}$$

where $K = 8\pi G/c^4$ and G is the gravitational constant. This strengthens our conjecture that the geometry of the spacetime continuum can be seen as a representation of the deformation of the spacetime continuum resulting from the strains generated by the energy-momentum stress tensor.

Rest-Mass Energy Relation

As shown in [1], the contraction of Eq.(4) yields the relation

$$2(\mu_0 + 2\lambda_0)\varepsilon = T^{\alpha}{}_{\alpha} \equiv T \tag{7}$$

where $T^{\alpha}{}_{\alpha}$ corresponds to the invariant rest-mass energy density

$$T^{\alpha}{}_{\alpha} = T = \rho c^2 \tag{8}$$

where ρ is the rest-mass density. The relation between the invariant volume dilatation ε and the invariant rest-mass energy density is thus given by

$$2(\mu_0 + 2\lambda_0)\varepsilon = \rho c^2 \tag{9}$$

or, in terms of the bulk modulus κ_0,

$$4\kappa_0 \varepsilon = \rho c^2. \tag{10}$$

As we noted in [1], this equation demonstrates that rest-mass energy density arises from the volume dilatation of the spacetime continuum. The rest-mass energy is equivalent to the energy required to dilate the volume of the spacetime continuum, and is a measure of the energy stored in the spacetime continuum as volume dilatation. The volume dilatation is an invariant, as is the rest-mass energy density.

Decomposition into Distortions and Dilatations

As also shown in [1], when the strain tensor $\varepsilon^{\mu\nu}$ and the energy-momentum stress tensor $T^{\mu\nu}$ are decomposed into a deviation tensor (the *distortion*) and a scalar (the *dilatation*), the strain-stress relation then becomes separated into dilatation and distortion relations:

$$\begin{aligned} &\text{dilatation}: t = 2(\mu_0 + 2\lambda_0)e = 4\kappa_0 e = \kappa_0 \varepsilon \\ &\text{distortion}: t^{\mu\nu} = 2\mu_0 e^{\mu\nu} \end{aligned} \tag{11}$$

where

$$\varepsilon^{\mu\nu} = e^{\mu\nu} + e g^{\mu\nu} \tag{12}$$

with

$$e^{\mu}{}_{\nu} = \varepsilon^{\mu}{}_{\nu} - e\delta^{\mu}{}_{\nu} \tag{13}$$

$$e = \frac{1}{4}\varepsilon^{\alpha}{}_{\alpha} = \frac{1}{4}\varepsilon \tag{14}$$

and similarly

$$T^{\mu\nu} = t^{\mu\nu} + t g^{\mu\nu} \tag{15}$$

with

$$t^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} - t\delta^{\mu}{}_{\nu} \tag{16}$$

$$t = \frac{1}{4} T^{\alpha}{}_{\alpha}. \tag{17}$$

The distortion-dilatation decomposition is evident in the dependence of the dilatation relation on the bulk modulus κ_0 and of the distortion relation on the shear modulus μ_0. The dilatation relation of Eq.(11) corresponds to rest-mass energy, while the distortion relation is traceless and thus massless, and corresponds to shear transverse waves. We also noted in [1] that this decomposition of spacetime continuum deformations into a massive dilatation and a massless transverse wave distortion is somewhat reminiscent of wave-particle duality.

3 Kinematic Relations

The strain $\varepsilon^{\mu\nu}$ can be expressed in terms of the displacement u^{μ} through the kinematic relation [6, see pp. 149–152]:

$$\varepsilon^{\mu\nu} = \frac{1}{2}(u^{\mu;\nu} + u^{\nu;\mu} + u^{\alpha;\mu}u_{\alpha}{}^{;\nu}) \tag{18}$$

where the semicolon (;) denotes covariant differentiation. For small displacements, this expression can be linearized to give the symmetric tensor

$$\varepsilon^{\mu\nu} = \frac{1}{2}(u^{\mu;\nu} + u^{\nu;\mu}) = u^{(\mu;\nu)}. \tag{19}$$

We use the small displacement approximation in this analysis.

An antisymmetric tensor $\omega^{\mu\nu}$ can also be defined from the displacement u^{μ}. This tensor is called the rotation tensor and is defined as [6]:

$$\omega^{\mu\nu} = \frac{1}{2}(u^{\mu;\nu} - u^{\nu;\mu}) = u^{[\mu;\nu]}. \tag{20}$$

Where needed, displacements in expressions derived from Eq.(19) will be written as $u_{\|}$ while displacements in expressions derived from Eq.(20) will be written as $u_{\perp}$. Using different symbolic subscripts for these displacements provides a reminder that symmetric displacements are along the direction of motion (longitudinal), while antisymmetric displacements are perpendicular to the direction of motion (transverse).

In general, we have [6]

$$u^{\mu;\nu} = \varepsilon^{\mu\nu} + \omega^{\mu\nu} \tag{21}$$

where the tensor $u^{\mu;\nu}$ is a combination of symmetric and antisymmetric tensors. Lowering index ν and contracting, we get the volume dilatation of the spacetime continuum

$$u^{\mu}{}_{;\mu} = \varepsilon^{\mu}{}_{\mu} = u_{\|}{}^{\mu}{}_{;\mu} = \varepsilon \tag{22}$$

where the relation

$$\omega^{\mu}{}_{\mu} = u_{\perp}{}^{\mu}{}_{;\mu} = 0 \tag{23}$$

has been used.

4 Dynamic Equation

4.1 Equilibrium Condition

Under equilibrium conditions, the dynamics of the spacetime continuum is described by the equation [4, see pp. 88–89],

$$T^{\mu\nu}{}_{;\mu} = -X^{\nu} \tag{24}$$

where X^{ν} is the volume (or body) force. As Wald [7, see p. 286] points out, in General Relativity the local energy density of matter as measured by a given observer is well-defined, and the relation

$$T^{\mu\nu}{}_{;\mu} = 0 \tag{25}$$

can be taken as expressing local conservation of the energy-momentum of matter. However, it does not in general lead to a global conservation law. The value $X^{\nu} = 0$ is thus taken to represent the macroscopic local case, while Eq.(24) provides a more general expression.

At the microscopic level, energy is conserved within the limits of the Heisenberg Uncertainty Principle. The volume force may thus be very small, but not exactly zero. It again makes sense to retain the volume force in the equation, and use Eq.(24) in the general case, while Eq.(25) can be used at the macroscopic local level, obtained by setting the volume force X^{ν} equal to zero.

4.2 Displacement Wave Equation

Substituting for $T^{\mu\nu}$ from Eq.(4), Eq.(24) becomes

$$2\mu_0\varepsilon^{\mu\nu}{}_{;\mu} + \lambda_0 g^{\mu\nu}\varepsilon_{;\mu} = -X^{\nu} \tag{26}$$

and, using Eq.(19),

$$\mu_0(u^{\mu;\nu}{}_{\mu} + u^{\nu;\mu}{}_{\mu}) + \lambda_0\varepsilon^{;\nu} = -X^{\nu}. \tag{27}$$

Interchanging the order of differentiation in the first term and using Eq.(22) to express ε in terms of u, this equation simplifies to

$$\mu_0 u^{\nu;\mu}{}_{\mu} + (\mu_0 + \lambda_0)u^{\mu}{}_{;\mu}{}^{\nu} = -X^{\nu} \tag{28}$$

which can also be written as

$$\mu_0\nabla^2 u^{\nu} + (\mu_0 + \lambda_0)\varepsilon^{;\nu} = -X^{\nu}. \tag{29}$$

This is the *displacement wave equation*.

Setting X^{ν} equal to zero, we obtain the macroscopic displacement wave equation

$$\nabla^2 u^{\nu} = -\frac{\mu_0 + \lambda_0}{\mu_0}\varepsilon^{;\nu}. \tag{30}$$

4.3 Continuity Equation

Taking the divergence of Eq.(21), we obtain

$$u^{\mu;\nu}{}_{\mu} = \varepsilon^{\mu\nu}{}_{;\mu} + \omega^{\mu\nu}{}_{;\mu}. \tag{31}$$

Interchanging the order of partial differentiation in the first term, and using Eq.(22) to express u in terms of ε, this equation simplifies to

$$\varepsilon^{\mu\nu}{}_{;\mu} + \omega^{\mu\nu}{}_{;\mu} = \varepsilon^{;\nu}. \tag{32}$$

Hence the divergence of the strain and rotation tensors equals the gradient of the massive volume dilatation, which acts as a source term. This is the continuity equation for deformations of the spacetime continuum.

5 Wave Equations

5.1 Dilatational (Longitudinal) Wave Equation

Taking the divergence of Eq.(28) and interchanging the order of partial differentiation in the first term, we obtain

$$(2\mu_0 + \lambda_0)u^{\mu}{}_{;\mu}{}^{\nu}{}_{\nu} = -X^{\nu}{}_{;\nu}. \tag{33}$$

Using Eq.(22) to express u in terms of ε, this equation simplifies to

$$(2\mu_0 + \lambda_0)\varepsilon^{;\nu}{}_{\nu} = -X^{\nu}{}_{;\nu} \tag{34}$$

or

$$(2\mu_0 + \lambda_0)\nabla^2\varepsilon = -X^{\nu}{}_{;\nu}. \tag{35}$$

Setting X^{ν} equal to zero, we obtain the macroscopic longitudinal wave equation

$$(2\mu_0 + \lambda_0)\nabla^2\varepsilon = 0. \tag{36}$$

The volume dilatation ε satisfies a wave equation known as the dilatational wave equation [6, see p. 260]. The solutions of the homogeneous equation are dilatational waves which are longitudinal waves, propagating along the direction of motion. Dilatations thus propagate in the spacetime continuum as longitudinal waves.

5.2 Rotational (Transverse) Wave Equation

Differentiating Eq.(28) with respect to x^α, we obtain

$$\mu_0 u^{\nu;\mu}{}_{\mu}{}^{\alpha} + (\mu_0 + \lambda_0) u^{\mu}{}_{;\mu}{}^{\nu\alpha} = -X^{\nu;\alpha}. \tag{37}$$

Interchanging the dummy indices ν and α, and subtracting the resulting equation from Eq.(37), we obtain the relation

$$\mu_0 (u^{\nu;\mu}{}_{\mu}{}^{\alpha} - u^{\alpha;\mu}{}_{\mu}{}^{\nu}) = -(X^{\nu;\alpha} - X^{\alpha;\nu}). \tag{38}$$

Interchanging the order of partial differentiations and using the definition of the rotation tensor $\omega^{\nu\alpha}$ of Eq.(20), the following wave equation is obtained:

$$\mu_0 \nabla^2 \omega^{\mu\nu} = -X^{[\mu;\nu]} \tag{39}$$

where $X^{[\mu;\nu]}$ is the antisymmetrical component of the gradient of the volume force defined as

$$X^{[\mu;\nu]} = \frac{1}{2}(X^{\mu;\nu} - X^{\nu;\mu}). \tag{40}$$

Setting X^ν equal to zero, we obtain the macroscopic transverse wave equation

$$\mu_0 \nabla^2 \omega^{\mu\nu} = 0. \tag{41}$$

The rotation tensor $\omega^{\mu\nu}$ satisfies a wave equation known as the rotational wave equation [6, see p. 260]. The solutions of the homogeneous equation are rotational waves which are transverse waves, propagating perpendicular to the direction of motion. Massless waves thus propagate in the spacetime continuum as transverse waves.

5.3 Strain (Symmetric) Wave Equation

A corresponding symmetric wave equation can also be derived for the strain $\varepsilon^{\mu\nu}$. Starting from Eq.(37), interchanging the dummy indices ν and α, adding the resulting equation to Eq.(37), and interchanging the order of partial differentiation, the following wave equation is obtained:

$$\mu_0 \nabla^2 \varepsilon^{\mu\nu} + (\mu_0 + \lambda_0) \varepsilon^{;\mu\nu} = -X^{(\mu;\nu)} \tag{42}$$

where $X^{(\mu;\nu)}$ is the symmetrical component of the gradient of the volume force defined as

$$X^{(\mu;\nu)} = \frac{1}{2}(X^{\mu;\nu} + X^{\nu;\mu}). \tag{43}$$

Setting X^ν equal to zero, we obtain the macroscopic symmetric wave equation

$$\nabla^2 \varepsilon^{\mu\nu} = -\frac{\mu_0 + \lambda_0}{\mu_0} \varepsilon^{;\mu\nu}. \tag{44}$$

This strain wave equation is similar to the displacement wave equation Eq.(30).

6 Discussion and Conclusion

In this paper, we have proposed a framework for the analysis of strained spacetime based on the elastodynamics of the spacetime continuum (*STCED*). In this model, the emphasis is on the displacements of the spacetime continuum infinitesimal elements from their unstrained configuration as a result of the strains applied on the *STC* by the energy-momentum stress tensor, rather than on the geometry of the *STC* due to the energy-momentum stress tensor.

We postulate that this description based on the deformation of the continuum is a description complementary to that of General Relativity which is concerned with modeling the resulting geometry of the spacetime continuum. Interestingly, the structure of the resulting stress-strain relation is similar to that of the field equations of General Relativity. This strengthens our conjecture that the geometry of the spacetime continuum can be seen as a representation of the deformation of the spacetime continuum resulting from the strains generated by the energy-momentum stress tensor. The equivalency of the strain description and of the geometrical description still remains to be demonstrated.

The equilibrium dynamic equation of the spacetime continuum is described by $T^{\mu\nu}{}_{;\mu} = -X^\nu$. In General Relativity, the relation $T^{\mu\nu}{}_{;\mu} = 0$ is taken as expressing local conservation of the energy-momentum of matter. The value $X^\nu = 0$ is thus taken to represent the macroscopic local case, while in the general case, the volume force X^ν is retained in the equation. This dynamic equation leads to a series of wave equations as derived in this paper: the displacement (u^ν), dilatational (ε), rotational ($\omega^{\mu\nu}$) and strain ($\varepsilon^{\mu\nu}$) wave equations.

Hence energy is seen to propagate in the spacetime continuum as deformations of the *STC* that satisfy wave equations of propagation. Deformations can be decomposed into dilatations and distortions. *Dilatations* involve an invariant change in volume of the spacetime continuum which is the source of the associated rest-mass energy density of the deformation. *Distortions* correspond to a change of shape of the spacetime continuum without a change in volume and are thus massless. Dilatations correspond to longitudinal displacements and distortions correspond to transverse displacements of the spacetime continuum.

Hence, every excitation of the spacetime continuum can be decomposed into a transverse and a longitudinal mode of propagation. We have noted that this decomposition into a dilatation with rest-mass energy density and a massless transverse wave distortion, is somewhat reminiscent of wave-particle duality, with the transverse mode corresponding to the wave aspects and the longitudinal mode corresponding to the particle aspects.

A continuity equation for deformations of the spacetime continuum is derived; we find that the divergence of the strain and rotation tensors equals the gradient of the massive volume dilatation, which acts as a source term.

The nature of the spacetime continuum volume force remains to be investigated. In addition, the displacement, dilatational, rotational and strain inhomogeneous wave equations need further investigation.

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