Change of Measure between Light Travel Time and Euclidean Distances

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The problem of cosmological distances is approached using a method based on the propagation of light in an expanding Universe. From the change of measure between Light Travel Time and Euclidean Distances, a formula is derived to compute distances as a function of redshift. This formula is identical to Mattig's formula (with $q_0 = 1/2$) which is based on Friedmann's equations of general relativity.

1 Introduction

Euclidean Distances were introduced in [1], and it was suggested that Euclidean Distances need to be used in order to derive the galactic density profile which is the evolution of galactic density over time. The LTD (Light Travel Distance) is the distance traversed by a photon between the time it is emitted and the time it reaches the observer, which may be also referred to as the Light Travel Time. We define the Euclidean distance as the equivalent distance that would be traversed by a photon between the time it is emitted and the time it reaches the observer if there were no expansion of the Universe.

In the present study, a time-varying Hubble coefficient in the Euclidean framework is introduced assuming that the Hubble law observed in the LTD framework is still applicable in the Euclidean framework. The model provides a "kinematic age of the Universe" which is purely mathematical as it is a result of the change of measure between LTDs and Euclidean Distances. A proof is made that a flat Hubble constant in the LTD framework (i.e that does not vary with LTD) is equivalent to a second order forward time-varying Euclidean Hubble coefficient in the Euclidean framework.

2 Foundations of the theory

The observed Hubble constant that is commonly referred to in the literature is a measure of space expansion with respect to LTDs. The Euclidean Hubble coefficient is being defined as the space expansion with respect to Euclidean Distances. This is a change of measure considering that the Euclidean Hubble coefficient varies with time such that the Hubble law is still applicable in the Euclidean framework. This leads to the following equation

$$H_i(t) = \frac{\dot{a}}{a}\,, \tag{1}$$

where H_i is the instantaneous Euclidean Hubble coefficient, $\dot{a}$ is the Universe expansion velocity and a the scale factor

The main postulate of the present study is that the Euclidean Hubble coefficient needs to be used in order to compute the scale factor in metric distances and not on the basis of LTDs, see (1). If we did not compute the scale factor on the basis of metric distances, the equation would fail to work with cosmological redshifts, which are a homothetic transformation for describing the evolution of light wavelength.

The instantaneous Euclidean Hubble coefficient is defined as the rate of expansion in Euclidean metrics at any given point in time along the trajectory of a light ray reaching the observer.

As space between the photon and the observer expands, this expansion is added to the overall distance the photon has to travel in order to reach the observer; therefore, the Euclidean Distance between the photon and the observer is defined by the following differential equations, respectively in the temporal and metric form:

1) In the LTD framework (the temporal form)

$$\frac{dy}{dt} = -c + H_0\, c\, T\,, \tag{2}$$

where: y is the Euclidean Distance between the photon and the observer, T the LTD between the observer and the photon, c the celerity of light, and H_0 the Hubble constant as of today;

2) In the Euclidean framework (the metric form)

$$\frac{dy}{dt} = -c + H_i(t)\, y\,, \tag{3}$$

where y is the Euclidean Distance between the photon and the observer, c the celerity of light, and $H_i(t)$ the Euclidean time-varying Hubble coefficient.

For the purpose of convenience let us consider the following form for the Euclidean time-varying Hubble coefficient

$$H_i(t) = \frac{n}{t}\,, \tag{4}$$

where $H_i(t)$ is the Euclidean time-varying Hubble coefficient, n the order of the time-varying Euclidean Hubble, and t the time from the hypothetical big bang for which time was set to zero.

Note that in the present study both the Hubble constant and the Euclidean Hubble coefficient are expressed in units of $[time^{-1}]$ by converting all distances into Light Travel Time, and with the celerity $c = 1$.

3 Proof that a flat Hubble constant in the LTD framework is time varying of order two in the Euclidean framework

First, let us solve the differential equation for the propagation of light in the LTD framework assuming a flat Hubble constant (i.e. that does not vary with LTD). Let us consider a photon initially situated at a Euclidean Distance y_0 from the observer and moving at celerity c in the direction of the observer. Let us say T is the initial LTD between the photon and the observer, and define the Hubble constant function of LTDs.

The differential equation describing the propagation of light in the LTD framework is described by (2). By setting time zero at a reference T_b in the past, we have $t = T_b - T$; therefore, $dt = -dT$. Hence, (2) becomes

$$\frac{dy}{dT} = c - H_0\, c\, T\,, \qquad (5)$$

with boundary conditions $y(T) = y_0$, and $y(0) = 0$.

By integration from 0 to T, the following relationship relating Euclidean Distances y to Light Travel Distances T is obtained

$$y = c\,T - \frac{c\,H_0\,T^2}{2}\,. \qquad (6)$$

Now let us derive the differential equation for the propagation of light in the Euclidean framework assuming the time-varying Hubble coefficient from (4) (see Figure 1). From the differential equation describing the propagation of light in the Euclidean framework (3), we get

$$\frac{dy}{dt} = -c + \frac{n}{t}\,y\,. \qquad (7)$$

By integrating this first order non-homogeneous differential equation between $T_b - T$ and T_b, the following solution is obtained which describes the relationship between Euclidean Distances and LTDs

$$y = \frac{c}{n-1}\left(Tb - T - Tb^{1-n}\,(Tb - T)^n\right). \qquad (8)$$

By setting n equal to 2 in (8) for a second order time-varying Hubble coefficient, we get

$$y = c\left(T - \frac{T^2}{T_b}\right). \qquad (9)$$

Based on the recession speed, the relationship between the Hubble constant defined function of LTDs, and the Euclidean Hubble, for T small is as follows

$$H_0\, c\, T = \frac{n}{(Tb - T)}\,y\,. \qquad (10)$$

Hence, $\frac{n}{T_b}$ is obtained by computing the following limit

$$\frac{n}{T_b} = \lim_{T\to 0}\left(\frac{H_0\, c\, T}{y}\right). \qquad (11)$$

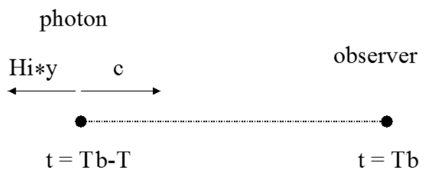

Fig. 1: Schema to represent the propagation of light in an expanding space in the Euclidean framework. Where T is the Light Travel Distance between the observer and the source of light, T_b is the kinematic age of the Universe, and n the order of the time-varying Hubble coefficient (time zero set at time T_b from today).

By substitution of y from (8), we get

$$\frac{n}{T_b} = \lim_{T\to 0}\left(\frac{(n-1)\,T\cdot H_0}{T_b - T - T_b^{1-n}\,(T_b - T)^n}\right) = H_0\,. \qquad (12)$$

Therefore, the "kinematic age of the Universe" is

$$T_b = \frac{n}{H_0}, \qquad (13)$$

with H_0 the Hubble constant as of today.

By substitution of $T_b = \frac{2}{H_0}$ into (9), we get

$$y = c\,T - \frac{c\,H_0\,T^2}{2}\,. \qquad (14)$$

This solution is identical to (6) relating LTDs to Euclidean Distances for the flat Hubble constant in the LTD framework. This is the proof that a flat Hubble constant in the LTD framework is equivalent to a time-varying Hubble coefficient of order two in the Euclidean framework. The equation $H_i(t) = 2/t$ is the connection between (2) and (3).

We can easily show that the recession speed with the second order time-varying Hubble coefficient in the Euclidean framework is the same as the recession speed in the LTD framework. The calculations are as follow

$$H_i(t)\; y = \frac{2}{t}\,y = \frac{2\,c}{T_b - T}\left(T - \frac{H_0\,T^2}{2}\right). \qquad (15)$$

By substitution of T_b from (13) (with a second order time-varying Hubble coefficient) into (15), we obtain

$$H_i(t)\,y = H_0\, c\, T\,, \qquad (16)$$

where T is the LTD between the observer and the source of light, and y the Euclidean Distance.

4 Evolutionary model of the scale factor

The differential equation describing the evolution of the scale factor a is as follows, identical to (1),

$$\frac{da}{dt} = H_i(t)\,a\,. \qquad (17)$$

As $H_i(t) = \frac{2}{t}$, we get

$$\int_{a_1}^{a_0} \frac{1}{a} da = \int_{T_b - T}^{T_b} \frac{2}{t} dt . \tag{18}$$

By integrating (18) , we obtain

$$\ln\left(\frac{a_0}{a_1}\right) = 2 \ln\left(\frac{T_b}{T_b - T}\right), \tag{19}$$

which is equivalent to

$$\frac{a_0}{a_1} = \left(\frac{T_b}{T_b - T}\right)^2 . \tag{20}$$

5 Expression of distances versus redshifts

From cosmological redshifts, we have

$$1 + z = \frac{a_0}{a_1} , \tag{21}$$

where a_0 is the present scale factor, a_1 the scale factor at redshift z.

Combining (20) and (21), we get

$$T = T_b \left(1 - \frac{1}{\sqrt{1+z}}\right). \tag{22}$$

By substitution of T_b from (13) for a second order time-varying Hubble coefficient, we get the following equation relating LTD to redshifts

$$T = \frac{2}{H_0} \left(1 - \frac{1}{\sqrt{1+z}}\right). \tag{23}$$

6 Comparison with the equation of Mattig

The equation of Mattig [2] is as follows

$$rR_0 = \frac{1}{H_0 q_0^2 (1+z)} \times \times \left(q_0 z + (q_0 - 1)(\sqrt{1 + 2q_0 z} - 1)\right), \tag{24}$$

where r is the distance, q_0 is the deceleration parameter, R_0 the present scale factor, z the redshift, H_0 the present scale factor.

For comparison purpose with the equation of the present study, we should set q_0 equal to 1/2 (flat matter dominated Universe), and R_0 to 1. Therefore, we obtain

$$r = \frac{2}{H_0} \left(1 - \frac{1}{\sqrt{1+z}}\right). \tag{25}$$

This formula is identical to (23). We have just shown that the solution to our problem is identical to Mattig formula for q_0 equal to 1/2.

7 Discussion

Based on the change of measure between LTD and Euclidean Distances, a formula that expresses distances versus redshifts is obtained. From the change of framework between LTD and Euclidean distances, it has been proved that a flat Hubble constant (that does not vary with LTD) is equivalent to a time-varying Euclidean Hubble coefficient of order two. Finally, the evolutionary model of the scale factor is derived and matched to the cosmological redshift equation in order to obtain the LTD versus redshift equation. This equation is identical to Mattig's formula (with $q_0 = 1/2$) which is based on Friedmann's equations of general relativity. The Euclidean Hubble coefficient was used in order to derive the evolution of the scale factor in metric distances; otherwise, the cosmological redshift equation would not be applicable to light wavelengths. This study proposes a new approach to compute cosmological distances which is based on the introduction of Euclidean Distances in addition to Light Travel Distances in an expanding Universe, and a change of measure. The calculations involved are quite simple and our definition of Euclidean Distances may be used as a source of inspiration to develop future cosmological models.

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References

1. Heymann Y. Building galactic density profiles. *Progress in Physics*, 2011, v. 4, 63–67.
2. Mattig W. Über den Zusammenhang zwischen Rotverschiebung und scheinbarer Helligkeit. *Astronomische Nachrichten*, 1958, v. 284, 109–111.