Flow of Viscous Fluid between Two Parallel Porous Plates with Bottom Injection and Top Suction

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This paper deals with the problem of steady laminar flow of viscous incompressible fluid between two parallel porous plates with bottom injection and top suction. The flow is driven by a pressure gradient $\frac{\partial p}{\partial x}$ and uniform vertical flow is generated i.e. the vertical velocity is constant everywhere in the field flow i.e. $v = v_w = constant$. Also a solution for the small and large Reynold number is discussed and the graph of velocity profile for flow between parallel plates with the bottom injection and top suction for different values of Reynold numbers is drawn.

1 Introduction

The two dimensional steady laminar flow in channels with porous walls has numerous application in field of Science and Engineering through boundary layer control, transpiration cooling and biomedical enginering.

Berman (1953) was the first reasercher who studied the problem of steady flow in an incompressible viscous fluid through a porous channel with rectangular cross section, when the Reynold number is low and the pertubation solution assuming normal wall velocity to be equal was obtained [1].

Sellars (1955), extended problem studied by Berman by using very high Reynold numbers [2].

Yuan (1956) [3] and Terill (1964) [4] analysed the same problem assuming different normal velocity at the wall.

Terrill and Shrestha (1965) analysed the problem for a flow in a channel with walls of different permeabilities [5].

Green (1979) studied the flow in a channel with one porous wall [7].

In this paper, we considered the flow of an incompressible viscous fluid between two parallel porous plates with bottom injection and top suction and assume that the wall velocity is uniform.

2 Formulation of the problem

The study laminar flow of an incompressible viscous fluid between two parallel porous plates with bottom injection and top suction at walls and uniform cross flow velocity is considered. The well known governing equations of the flow are:

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1}$$

Momentum equations (without body force):

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \tag{3}$$

The flow between two porous plates at y=+h and y=-h, respectively is considered.THe flow is deriven by a pressure gradient $\frac{\partial p}{\partial x}$. It is assumeed that a uniform vertical flow is generated i.e the vertical velocity component is constant everywhere in the flow field i.e $v = v_w = constant$. Again the continuity equation shows that $u = u(y)$ only, the momentum equation (2) becomes:

$$v_w\frac{du}{dy} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{d^2u}{dy^2}. \tag{4}$$

Re-arranging eqn. (4), we have

$$\frac{d^2u}{dy^2} - \frac{v_w}{\nu}\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}. \tag{5}$$

Homogeneous part of eqn. (5) becomes

$$\frac{d^2u}{dy^2} - \frac{v_w}{\nu}\frac{du}{dy} = 0. \tag{6}$$

Eqn. (6) is differential equation, with auxiliary equation of

$$p^2 - \frac{v_w}{\nu}p = 0$$

with roots

$$p_1 = 0, p_2 = \frac{v_w}{\nu}.$$

The solution of eqn. (6) is of the form

$$u(y) = Ae^{p_1 y} + Be^{p_2 y},$$

where A and B are constant.

$$u(y) = A + Be^{\frac{v_w}{\nu}y} \tag{7}$$

For particular integral of eqn. (5), we set

$$u(y) = ay^2 + by + c, \tag{8}$$

where a, b, and c are constants.

$$\frac{du}{dy} = 2ay + b, \Rightarrow \frac{d^2u}{dy^2} = 2a \tag{9}$$

Substituting eqn. (9) in eqn. (5) we get

$$\left(2a - \frac{v_w}{\nu}b\right) - 2a\frac{v_w}{\nu}y = \frac{1}{\mu}\frac{dp}{dx}.$$

Comparing the co-efficients, we get

$$a = 0 \Rightarrow b = -\frac{\nu}{v_w}\frac{1}{\mu}\frac{dp}{dx}. \tag{10}$$

Now, eqn. (8) becomes

$$u(y) = -\frac{\nu}{v_w}\frac{1}{\mu}\frac{dp}{dx}y + c. \tag{11}$$

The final solution formes by adding eqn. (7) and eqn. (11)

$$u(y) = D + Be^{\frac{v_w}{\nu}y} - \frac{\nu}{v_w}\frac{1}{\mu}\frac{dp}{dx}y. \tag{12}$$

Since v_w is constant, the equation is linear. We retain the no-slip condition for the main flow.

$$u(+h) = u(-h) = 0$$

$$u(h) = D + Be^{\frac{v_w}{\nu}h} - \frac{\nu}{v_w}\frac{1}{\mu}\frac{dp}{dx}h \tag{13}$$

$$u(-h) = D + Be^{-\frac{v_w}{\nu}h} + \frac{\nu}{v_w}\frac{1}{\mu}\frac{dp}{dx}h. \tag{14}$$

Subtracting eqn. (14) from eqn. (13), we get

$$B = \frac{2\frac{\nu}{v_w}\frac{h}{\mu}\frac{dp}{dx}}{e^{\frac{v_w}{\nu}}h - e^{-\frac{v_w}{\nu}}h} = \frac{2\frac{\nu}{v_w}\frac{h}{\mu}\frac{dp}{dx}}{2\sinh\left(\frac{v_w}{\nu}h\right)} = \frac{\frac{\nu}{v_w}\frac{h}{\mu}\frac{dp}{dx}}{\sinh\left(\frac{v_w}{\nu}h\right)}. \tag{15}$$

Substituting eqn. (15) into eqn. (13), we get

$$D = -\frac{\frac{\nu}{v_w}\frac{h}{\mu}\frac{dp}{dx}e^{\frac{v_w}{\nu}h}}{\sinh\left(\frac{v_w}{\nu}h\right)} + \frac{\nu}{v_w}\frac{h}{\mu}\frac{dp}{dx}. \tag{16}$$

Eqn. (12) reduces to

$$u(y) = -\frac{\frac{\nu}{v_w}\frac{h}{\mu}\frac{dp}{dx}e^{\frac{v_w}{\nu}h}}{\sinh\left(\frac{v_w}{\nu}h\right)} + \frac{\nu}{v_w}\frac{h}{\mu}\frac{dp}{dx} + \frac{\frac{\nu}{v_w}\frac{h}{\mu}\frac{dp}{dx}e^{\frac{v_w}{\nu}y}}{\sinh\left(\frac{v_w}{\nu}h\right)} - \frac{\nu}{v_w}\frac{y}{\mu}\frac{dp}{dx}. \tag{17}$$

But wall Reynold number is $Re = \frac{v_w}{\nu}h$, $\frac{Re}{h} = \frac{v_w}{\nu} \Rightarrow \frac{h}{Re} = \frac{\nu}{v_w}$.

Re-arranging eqn. (17), we get

$$u(y) = -\frac{h^2}{Re}\frac{1}{\mu}\frac{dp}{dx}\left[\frac{y}{h} - 1 + \frac{e^{Re} - e^{Re\frac{y}{h}}}{\sinh(Re)}\right]. \tag{18}$$

The final solution of eqn. (5),

$$\frac{u(y)}{u_{max}} = \frac{2}{Re}\left[\frac{y}{h} - 1 + \frac{e^{Re} - e^{Re\frac{y}{h}}}{\sinh Re}\right]. \tag{19}$$

Where $u_{max} = \frac{h^2}{2\mu}(-\frac{dp}{dy})$ is the centerline velocity for imporous or poiseuille.

For very small Re (or small vertical velocity), then the last terms in the parentheses of of eqn. (19) can be expanded in a power series and $\sinh Re \approx Re$ i.e.

$$\frac{u(y)}{u_{max}} = \frac{2}{Re}\left[\frac{y}{h} - 1 + \frac{1 + Re + \frac{(Re)^2}{2} + .. - \left(1 + Re\frac{y}{h} + \frac{(Re)^2}{2}\frac{y^2}{h^2} + ..\right)}{Re}\right],$$

$$\frac{u(y)}{u_{max}} = \frac{2}{Re}\left[\frac{y}{h} - 1 + \frac{Re\left(1 + \frac{Re}{2} - \frac{y}{h} - \frac{Re}{2}\frac{y^2}{h^2}\right)}{Re}\right],$$

$$\frac{u(y)}{u_{max}} = 1 - \frac{y^2}{h^2}. \tag{20}$$

Eqn. (20) shows that, the poiseuille solution recovered.

For very large Re (or large vertical velocity), eqn. (19) can be written as

$$\frac{u(y)}{u_{max}} = \frac{2}{Re}\left[\frac{y}{h} - 1 + 2\frac{e^{Re} - e^{Re\frac{y}{h}}}{e^{Re} - e^{-Re}}\right],$$

$$\frac{u(y)}{u_{max}} = \frac{2}{Re}\left[\frac{y}{h} - 1 + 2\frac{1 - e^{-Re\left(1 - \frac{y}{h}\right)}}{1 - e^{-2Re}}\right].$$

For $Re \to \infty$ and $\frac{y}{h} > 1$, except for $y = +h$, we get

$$\frac{u(y)}{u_{max}} = \frac{2}{Re}\left[\frac{y}{h} - 1 + 2\right],$$

$$\frac{u(y)}{u_{max}} = \frac{2}{Re}\left[1 + \frac{y}{h}\right], \tag{21}$$

so that a straight line variation which suddenly drops to zero at the upper wall.

3 Discussion

The velocity profiles have been drawn for different values of Reynold numbers (i.e. Re = 0, 3, 5, 10). From Fig. (1), its observed that for $Re \geq 0$ in the region $-1 \leq y^* \leq 1$, the shapes change smoothly with Reynold numbers and the average velocity is decreasing and Reynold number increases; i.e. the friction factor increases as we apply more cross flow through the wall.

4 Conclussion

In the above analysis a class of solution of flow of viscous fluid between two parallel porous plates with bottom injection and top suction is presented when a cross flow velocity along the boundary is uniform, the convective acceleration is linear and the flow is deriven from pressure gradient.

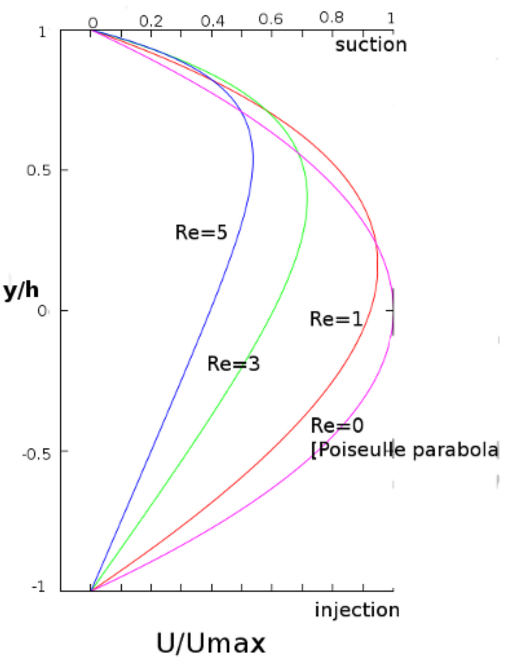

Fig. 1: Velocity profiles for flow between parallel plates with bottom injection and top suction for different values of Re.

Nomenclature

A,B,C,D: Constants
h: Height of the channel
P: Pressure
x: Axial distance
y:Lateral distance
v_w: Lateral wall velocity
u(x,y): Axial velocity component
v(x,y): Lateral velocity component
$y^* = \frac{y}{h}$: Dimensionless lateral distance
$Re = \frac{v_w h}{\nu}$: Wall Reynold number

Greek Symbols

μ: Shear viscosity
ν: Kinematic viscosity
ρ: Fluid density

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