Orbits in Homogeneous Time Varying Spherical Spacetime

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The solution to Einstein's gravitational field equations exterior to time varying distributions of mass within regions of spherical geometry is used to study the behaviour of test particles and photons in the vicinity of the mass distribution. Equations of motion are derived and an expression for deflection of light in this gravitational field is obtained. The expresion obtained differs from that in Schwarzschild's field by a multiplicative time dependent factor. The concept of gravitational lens in this gravitational field is also studied.

1 Introduction

In [1], the covariant metric tensor exterior to a homogeneous time varying distribution of mass within regions of spherical geometry is defined as:

$$g_{00} = -\left[1 + \frac{2}{c^2} f(t,r)\right] \tag{1}$$

$$g_{11} = \left[1 + \frac{2}{c^2} f(t,r)\right]^{-1} \tag{2}$$

$$g_{22} = r^2 \tag{3}$$

$$g_{33} = r^2 \sin^2\theta \tag{4}$$

where $f(t,r)$ is a function dependent on the mass distribution within the sphere that experiences radial displacement. Einstein's gravitational field equations were constructed in [1] and an approximate expression for the analytical solution of the lone field equation was obtained as

$$f(t,r) \approx -\frac{k}{r} \exp i\omega\left(t - \frac{r}{c}\right) \tag{5}$$

where $k = GM_0$ with G being the universal gravitational constant and M_0 the total mass of the spherical body. ω is the angular frequency of the radial displacement of mass within the sphere.

In this article, we use this solution of Einstein's field equations to study the behaviour of light in the vicinity of a time varying spherical mass distribution.

2 Orbits in Time Varying Spherical Spacetime

In order to study the motion of planets and light rays in a homogeneous time varying spherical spacetime, there is need to derive the geodesic equations [2]. The Lagrangian (L) for this gravitational field can be defined using the metric tensor as:

$$L=\frac{1}{c}\left[-g_{00}\left(\frac{dt}{d\tau}\right)^2 - g_{11}\left(\frac{dr}{d\tau}\right)^2 - g_{22}\left(\frac{d\theta}{d\tau}\right)^2 - g_{33}\left(\frac{d\phi}{d\tau}\right)^2\right]^{\frac{1}{2}} \tag{6}$$

Assuming that the orbits remain permanently in the equatorial plane (as in Newtonian Theory), then $\theta = \frac{\pi}{2}$ and the Lagrangian reduces to

$$L = \frac{1}{c}\left[-g_{00}\left(\frac{dt}{d\tau}\right)^2 - g_{11}\left(\frac{dr}{d\tau}\right)^2 - g_{33}\left(\frac{d\phi}{d\tau}\right)^2\right]^{\frac{1}{2}} \tag{7}$$

or more explicitly as

$$L = \frac{1}{c}\left[\left(1 + \frac{2}{c^2} f(t,r)\right)\dot{t}^2 - \left(1 + \frac{2}{c^2} f(t,r)\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2\right]^{\frac{1}{2}} \tag{8}$$

where the dot denotes differentiation with respect to proper time (τ).

Now, using the Euler-Lagrange equations and considering the fact that in a gravitational field is a conservative field, it can be shown that the law of conservation of energy in this field is given as

$$\left(1 + \frac{2}{c^2} f(t,r)\right)\dot{t} = d\,(constant) \tag{9}$$

or more explicitly as

$$\left[1 - \frac{2GM}{rc^2} \exp i\omega\left(t - \frac{r}{c}\right)\right]\dot{t} = d \tag{10}$$

which differs from that in Schwarzschild's field by the exponential factor that describes the radial displacement of mass with time.

It can also be shown that the law of conservation of angular momemtum in this gravitational field is given as

$$r^2\dot{\phi} = h\,(constant) \tag{11}$$

which is the same as that in Schwarzschild's field.

Let $L = \varepsilon$, and equation (8) becomes

$$\varepsilon^2=\left(1+\frac{2}{c^2}f(t,r)\right)\dot{t}^2-\frac{1}{c^2}\left[\left(1+\frac{2}{c^2}f(t,r)\right)^{-1}\dot{r}^2-r^2\dot{\phi}^2\right]. \tag{12}$$

Substituting equation (10) in (12) yields

$$\frac{1}{2}\left[\dot{r}^2+r^2\dot{\phi}^2\left(1+\frac{2}{c^2}f(t,r)\right)\right]+\varepsilon^2 f(t,r)=\frac{1}{2}c^2\left(d^2-\varepsilon^2\right). \tag{13}$$

This is the Newtonian energy equation with a modification to the $\dot{\phi}^2$ term. It is similar to that obtained in Schwarzschild's field except for the time dependent radial diplacement. Also, using equation (11), it can be shown that

$$\dot{r} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \dot{\phi}\frac{dr}{d\phi} = \frac{h}{r^2}\frac{dr}{d\phi}. \quad (14)$$

Now, let $u(\phi) = \frac{1}{r(\phi)}$ then

$$\dot{r} = -h\frac{du}{d\phi}. \quad (15)$$

Substituting equation (5) and (15) into equation (13) yields

$$\left(\frac{du}{d\phi}\right)^2 + u^2\left[1 - \frac{2k}{c^2}u \exp i\omega\left(t - \frac{1}{uc}\right)\right] + \frac{2\varepsilon^2 k}{h^2}u \exp i\omega\left(t - \frac{1}{uc}\right) = \frac{c^2}{h^2}\left(d^2 - \varepsilon^2\right) \quad (16)$$

It is worthnoting that integrating equation (16) directly leads to elliptical integrals which are ackward to handle; thus differentiating yields the following second order differential equation

$$\frac{d^2u}{d\phi^2} + u\left[1 - \frac{2k}{c^2}u \exp i\omega\left(t - \frac{1}{uc}\right)\right] - \frac{2k}{c^2}u^2\left(1 - \frac{1}{u}\right) \exp i\omega\left(t - \frac{1}{uc}\right) + \frac{2k\varepsilon^2}{h^2}\left(1 + \frac{1}{u^2}\right) \exp i\omega\left(t - \frac{1}{uc}\right) = 0. \quad (17)$$

This equation has additional terms not found in Schwarzschild's field.

3 Timelike Orbits and Precession

For timelike orbits $\varepsilon = 1$ and equation (17) becomes

$$\frac{d^2u}{d\phi^2} + u\left[1 - \frac{2k}{c^2}u \exp i\omega\left(t - \frac{1}{uc}\right)\right] - \frac{2k}{c^2}u^2\left(1 - \frac{1}{u}\right) \exp i\omega\left(t - \frac{1}{uc}\right) + \frac{2k}{h^2}\left(1 + \frac{1}{u^2}\right) \exp i\omega\left(t - \frac{1}{uc}\right) = 0. \quad (18)$$

Now as a first approximation, suppose $uc \gg 1$ and $k \ll h^2u^2$ then equation (8) reduces to

$$\frac{d^2u}{d\phi^2} + u = k\left[\frac{3}{c^2}u^2 + \frac{1}{c^2}u - \frac{1}{h^2}\right] \exp i\omega t. \quad (19)$$

The Newtonian equation for a spherical mass is

$$\frac{d^2u}{d\phi^2} + u = \frac{k}{h^2} \quad (20)$$

and that obtained in Schwarzschild's field is

$$\frac{d^2u}{d\phi^2} + u = \frac{k}{h^2} + \frac{3k}{c^2}u^2. \quad (21)$$

Apart from the first and second terms of equation (19) that are similar to Newton's equation and that in Schwarzschild's field, the other terms have terms dependent on the time rate of rotation of the mass content within the sphere [3].

Solution of the Newtonian equation (20) yields the well known conics

$$u_0 = \frac{1}{l}\left(1 + e\cos\theta\right) \quad (22)$$

where $l = \frac{h^2}{GM}$ and e is the eccentricity of the orbit. Attempting an approximate solution for equation (19) by substituting the Newtonian solution into the quadratic term in u on the right hand side and neglecting the term in u, a particular integral u_1 satisfies equation (19) such that

$$\frac{d^2u_1}{d\phi^2} + u_1 = k\left[\frac{3}{l^2c^2}\left(1 + e\cos\theta\right)^2 - \frac{1}{h^2}\right] \exp i\omega t. \quad (23)$$

Now suppose u_1 takes the form:

$$u_1 = A + B\phi\sin\phi + C\cos 2\phi \quad (24)$$

where A, B and C are constants, then it can be shown that

$$u_1 = \frac{k}{c^2}\left(\frac{3}{l^2} - \frac{1}{l} - \frac{1}{h^2}\right) \exp i\omega t + \frac{ke\phi}{2c^2}\left(\frac{3}{l^2} - \frac{1}{2l}\right)\sin 2\phi \exp i\omega t + \frac{ke^2}{l^2c^2}\cos 2\phi. \quad (25)$$

Then the approximate solution for u can be given as

$$u = u_0 + u_1 \quad (26)$$

or

$$u = \frac{1}{l}\left(1 + e\cos\theta\right) + \frac{k}{c^2}\left(\frac{3}{l^2} - \frac{1}{l} - \frac{1}{h^2}\right) \exp i\omega t + \frac{ke\phi}{2c^2}\left(\frac{3}{l^2} - \frac{1}{2l}\right)\sin 2\phi \exp i\omega t + \frac{ke^2}{l^2c^2}\cos 2\phi. \quad (27)$$

Hence, this approximate solution introduces corrections to u_0 and hence depicts that the orbits of massive objects is only approximately elliptical and also accounts for the perihelion precession of planetary orbits in this gravitational field.

4 The Bending of Light

For null geodesics, $\varepsilon = 0$ and equation (17) yields

$$\frac{d^2u}{d\phi^2} + u = \left[\frac{3k}{c^2} \exp i\omega\left(t - \frac{1}{uc}\right)\right]u^2 + \left[\frac{k}{c^2} \exp i\omega\left(t - \frac{1}{uc}\right)\right]u. \quad (28)$$

In the limit of Special Relativity, equation (28) reduces to

$$\frac{d^2u}{d\phi^2} + u = 0. \tag{29}$$

The general solution of equation (29) is given as

$$u = \frac{1}{b}\sin(\phi - \phi_0) \tag{30}$$

where b is the closest approach to the origin (or impact parameter). This is the equation of a straight line as ϕ goes from ϕ_0 to $\phi_0 + \pi$. The straight line motion of light is the same as that predicted by Newtonian theory.

Now, solving the General Relativity problem (equation 28) by taking the general solution (u) to be a pertubation of the Newtonian solution, and setting $\phi_0 = 0$, then

$$u = u_0 + u_1 \tag{31}$$

where $u_0 = \frac{1}{b}\sin\phi$. Thus, u_1 satisfies the equation

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{3k}{b^2c^2}\sin^2\phi \, \exp i\omega\left(t - \frac{b}{c\sin\phi}\right) + \frac{k}{bc^2}\sin\phi \, \exp i\omega\left(t - \frac{b}{c\sin\phi}\right). \tag{32}$$

Now, by considering a particular integral of the form

$$u_1 = A + B\sin^2\phi \tag{33}$$

and substituting in equation (32), it can be shown that

$$u_1 = \frac{2k}{b^2c^2}\left(1 - \frac{1}{2}\sin^2\phi\right)\exp i\omega\left(t - \frac{b}{c\sin\phi}\right) \tag{34}$$

and thus

$$u = \frac{1}{b}\sin\phi + \frac{2k}{b^2c^2}\left(1 - \frac{1}{2}\sin^2\phi\right)\exp i\omega\left(t - \frac{b}{c\sin\phi}\right). \tag{35}$$

Now, consider the deflection of a light ray from a star which just grazes the time varying homogeneous spherical mass (such as the Sun approximately); as in Fig. 1, then as $r \to \pm\infty$, $u \to 0$, so

$$0 = \frac{1}{b}\sin\phi + \frac{2k}{b^2c^2}\left(1 - \frac{1}{2}\sin^2\phi\right)\exp i\omega\left(t - \frac{b}{c\sin\phi}\right). \tag{36}$$

At the asymptotes, $\phi = -\psi_1$ and $\phi = \psi_2 + \pi$ and taking $\phi \ll 1$ then equation (36) reduces to

$$0 = \frac{1}{b}\psi_1 + \frac{2k}{b^2c^2}\exp i\omega\left(t + \frac{b}{c\psi_1}\right) \tag{37}$$

and

$$0 = \frac{1}{b}\psi_2 + \frac{2k}{b^2c^2}\exp i\omega\left(t + \frac{b}{c\psi_2}\right). \tag{38}$$

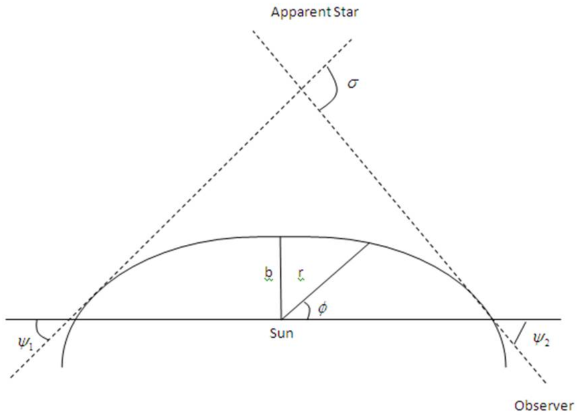

Fig. 1: Diagram showing the total deflection of light.

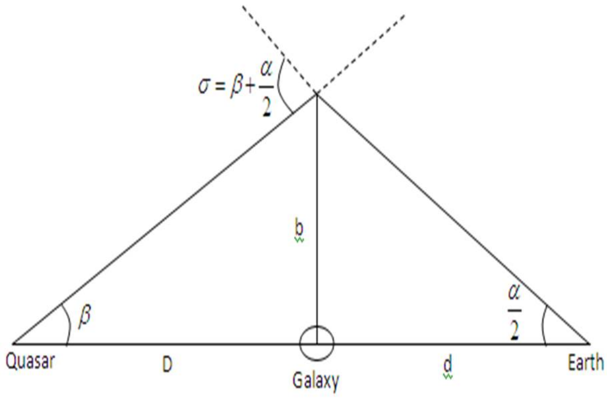

Fig. 2: Einstein's Ring.

The total deflection of light (σ) is given as

$$\sigma = \psi_1 + \psi_2$$

or

$$\sigma = \frac{2k}{bc^2}\left[\exp i\omega\left(t + \frac{b}{c\psi_1}\right) + \exp i\omega\left(t + \frac{b}{c\psi_2}\right)\right]. \tag{39}$$

Thus, the introduction of varying mass distribution with time introduces an exponential term in the deflection of light equation not found in static homogeneous spherical gravitational fields.

Now, as an example of the bending of light, let us consider a gravitational lens.

Consider a quasar directly behind a galaxy in our line of sight as shown in Fig. 2.

The distance of closest approach to the time varying spherical mass distribution corresponds to an angle (σ) given as equation (39). From Fig. 2, considering that both α and β are small, it can be deduced that

$$\sigma = \frac{\alpha}{2} + \beta = \frac{b}{d} + \frac{b}{D} \tag{40}$$

and substituting equation (39) yields the impact parameter as

$$b = \left\{ \frac{2k}{c^2} \left(\frac{Dd}{D+d} \right) \left[\exp\, i\omega \left(t + \frac{b}{c\psi_1} \right) + \exp\, i\omega \left(t + \frac{b}{c\psi_2} \right) \right] \right\}^{\frac{1}{2}} .$$

Hence, the image of the quasar appears as a ring which subtends an angle

$$\alpha = \frac{2b}{d}$$

or

$$\alpha = \frac{2}{c} \left\{ \frac{Dd}{d(D+d)} \left[\exp\, i\omega \left(t + \frac{b}{c\psi_1} \right) + \exp\, i\omega \left(t + \frac{b}{c\psi_2} \right) \right] \right\}^{\frac{1}{2}} .$$

5 Conclusion

The results obtained in this study has paved the way for the theoretical study of homogeneous spherical mass distributions in which the mass content is varying with time. This will introduce correction terms found in Schwarzschild's static field. It is hoped that using this approach experimentally and astrophysically more satisfactory expressions and values will be obtained for gravitational phenomena in the universe.

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