A Model for the Expansion of the Universe

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One introduces an ansatz for the expansion factor $a(t) = e^{(H(t)t-H_0T_0)/\beta}$ for our Universe in the spirit of the FLRW model; β is a constant to be determined. Considering that the ingredients acting on the Universe expansion ($t > 4\times10^{12}\ s \approx 1.3\times10^{-5}\ Gyr$) are mainly matter (baryons plus dark matter) and dark energy, one uses the current measured values of Hubble constant H_0, the Universe current age T_0, matter density parameter $\Omega_m(T_0)$ and dark energy parameter $\Omega_\Lambda(T_0)$ together with the Friedmann equations to find $\beta = 0.5804$ and that our Universe may have had a negative expansion acceleration up to the age $T_\star = 3.214\ Gyr$ (*matter era*) and positive after that (*dark energy era*), leading to an eternal expansion. An interaction between matter and dark energy is found to exist. The deceleration $q(t)$ has been found to be $q(T_\star) = 0$ and $q(T_0) = -0.570$.

1 Introduction

The Cosmological Principle states that the Universe is spatially homogeneous and isotropic on sufficiently large scale [1–4] and [7]. This is expressed by the Friedmann spacetime metric:

$$ds^2 = \Re^2(t)\, d\psi^2 + \Re^2(t) f_k^2(\psi)\left(d\theta^2 + sin^2\theta\, d\phi^2\right) - c^2dt^2, \quad (1)$$

where ψ, θ and ϕ are comoving space coordinates ($0 \le \psi \le \pi$, *for closed Universe*, $0 \le \psi \le \infty$, *for open and flat Universe*, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$), t is the proper time shown by any observer clock in the comoving system. $\Re(t)$ is the scale factor in units of distance; actually $\Re(t)$ is the *radius of curvature* of the Universe. The proper time t may be identified with the cosmic time. In terms of the usual expansion factor

$$a(t) = \frac{\Re(t)}{\Re(T_0)}, \quad (2)$$

being T_0 the current age of the Universe, equation (1) becomes

$$\begin{aligned} ds^2 = {} & \Re^2(T_0)\, a^2(t) \\ & \left(d\psi^2 + f_k^2(\psi)\left(d\theta^2 + sin^2\theta\, d\phi^2\right)\right) - c^2dt^2, \end{aligned} \quad (3)$$

$f_k^2(\psi)$ assumes the following expressions:

$$f_k^2(\psi)\begin{cases} f_1^2(\psi) = sin^2\psi \ \text{(closed Universe)} \\ f_0^2(\psi) = \psi^2 \ \text{(flat Universe)} \\ f_{-1}^2(\psi) = sinh^2\psi \ \text{(open Universe)} \end{cases} \quad (4)$$

The expansion process one will be considering here is the one started by the time of $4\times10^{12}\ s \approx 1.3\times10^{-5}\ Gyr$ when the so called *matter era* began. Right before that, the Universe went through the so called *radiation era*. In this paper one considers only the role of the matter (baryonic and non-baryonic) and the dark energy.

2 Einstein's field equations

Let one uses Einstein's Field Equations [5], with the inclusion of the Λ "cosmological constant" term.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4}\left(T_{\mu\nu} + T^{\Lambda}_{\mu\nu}\right) \quad (5)$$

where $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar curvature, $T_{\mu\nu}$ is the energy-momentum tensor, and, $T^{\Lambda}_{\mu\nu}$ the dark-energy-momentum tensor,

$$T^{\Lambda}_{\mu\nu} = \rho_\Lambda c^2 g_{\mu\nu}, \quad (6)$$

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}; \quad (7)$$

Λ is the "cosmological constant", which will be here allowed to vary with time. The metric tensor for the metric above, equation (3), is

$$(g_{\mu\nu}) = \begin{pmatrix} \Re^2(t) & 0 & 0 & 0 \\ 0 & \Re^2(t) f_k^2(\psi) & 0 & 0 \\ 0 & 0 & \Re^2(t) f_k^2(\psi) sin^2\theta & 0 \\ 0 & 0 & 0 & -c^2 \end{pmatrix} \quad (8)$$

where

$$\Re(t) = \Re(T_0)\, a(t). \quad (9)$$

The Ricci tensor is given by

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\eta\lambda} - \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\eta\nu} \quad (10)$$

where the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ are

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right). \quad (11)$$

The Ricci scalar curvature is given by

$$R = g^{\mu\nu} R_{\mu\nu}\,, \quad (12)$$

and the energy-momentum tensor is

$$T_{\mu\nu} = \left(\rho_m + \frac{1}{c^2} p_m\right) u_\mu u_\nu + p_m g_{\mu\nu} \,, \tag{13}$$

where ρ_m is the matter density and p_m is the matter pressure, both only time dependent. By making straightforward calculations, one gets

$$\begin{aligned} R &= 6\left(\frac{k}{\Re^2(T_0)a^2(t)} + \frac{1}{c^2}\left(\left(\frac{\dot{a}(t)}{a(t)}\right)^2 + \frac{\ddot{a}(t)}{a(t)}\right)\right) \\ &= 6\left(K(t) + \frac{1}{c^2}\left(\left(\frac{\dot{a}(t)}{a(t)}\right)^2 + \frac{\ddot{a}(t)}{a(t)}\right)\right). \end{aligned} \tag{14}$$

Here $K(t)$ is Gaussian curvature at cosmic time t:

$$K(t) = \frac{k}{\Re^2(t)} = \frac{k}{\Re^2(T_0)\, a^2(t)}. \tag{15}$$

The Einstein's field equations are

$$\begin{aligned} &G_{ii} = \frac{8\pi G}{c^4}\left(T_{ii} + T^{\Lambda}_{ii}\right) \leftrightarrow \\ &-\left(c^2 K(t) + \left(\frac{\dot{a}(t)}{a(t)}\right)^2 + 2\frac{\ddot{a}(t)}{a(t)}\right) = 8\pi G\left(\frac{1}{c^2}\, p_m - \rho_\Lambda\right) \end{aligned} \tag{16}$$

and

$$\begin{aligned} &G_{tt} = \frac{8\pi G}{c^4}\left(T_{tt} + T^{\Lambda}_{tt}\right) \leftrightarrow \\ &3\left(c^2 K(t) + \left(\frac{\dot{a}(t)}{a(t)}\right)^2\right) = 8\pi G\left(\rho_m + \rho_\Lambda\right) \end{aligned} \tag{17}$$

where $i = (\psi, \theta, \phi)$; all off-diagonal terms are null. The equation of state for dark energy is

$$p_\Lambda = -\rho_\Lambda c^2. \tag{18}$$

Simple manipulation of equations above leads to

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\left(\rho_m + 3\frac{1}{c^2} p_m - 2\rho_\Lambda\right), \tag{19}$$

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 + c^2 K(t) = \frac{8\pi G}{3}\left(\rho_m + \rho_\Lambda\right). \tag{20}$$

Equations (19-20) are known as Friedmann equations. Having in account that

$$\frac{\dot{a}(t)}{a(t)} = H(t), \tag{21}$$

$$\frac{\ddot{a}(t)}{a(t)} = \dot{H}(t) + H^2(t), \tag{22}$$

where $H(t)$ is time dependent Hubble parameter, and that pressure $p_m = 0$ (matter is treated as dust), one has

$$\dot{H}(t) + H^2(t) = \frac{8\pi G}{3}\left(-\frac{1}{2}\,\rho_m + \rho_\Lambda\right), \tag{23}$$

$$c^2 K(t) + H^2(t) = \frac{8\pi G}{3}\left(\rho_m + \rho_\Lambda\right), \tag{24}$$

or

$$\frac{\dot{H}(t)}{H^2(t)} + 1 = \frac{1}{\rho_{crit}}\left(-\frac{1}{2}\,\rho_m + \rho_\Lambda\right), \tag{25}$$

$$\frac{c^2 K(t)}{H^2(t)} + 1 = \frac{1}{\rho_{crit}}\left(\rho_m + \rho_\Lambda\right), \tag{26}$$

where

$$\rho_{crit} = \frac{3H^2(t)}{8\pi G} \tag{27}$$

is the so called critical density. From equations (25-26) one obtains, after simple algebra,

$$\rho_m = \frac{1}{4\pi G}\left(c^2 K(t) - \dot{H}(t)\right), \tag{28}$$

$$\rho_\Lambda = \frac{1}{4\pi G}\left(\frac{1}{2}\, c^2 K(t) + \frac{3}{2} H^2(t) + \dot{H}(t)\right), \tag{29}$$

or,

$$\Omega_m = \left(\frac{2}{3}\frac{c^2 K(t)}{H^2(t)} - \frac{2}{3}\frac{\dot{H}(t)}{H^2(t)}\right), \tag{30}$$

$$\Omega_\Lambda = \left(\frac{1}{3}\frac{c^2 K(t)}{H^2(t)} + \frac{2}{3}\frac{\dot{H}(t)}{H^2(t)} + 1\right), \tag{31}$$

where $\Omega_m = \rho_m/\rho_{crit}$ and $\Omega_\Lambda = \rho_\Lambda/\rho_{crit}$ are, respectively, the cosmological matter and dark energy density parameters.

The Ricci scalar curvature stands as

$$R = 6\left(K(t) + \frac{1}{c^2}\left(2H^2(t) + \dot{H}(t)\right)\right). \tag{32}$$

3 The ansatz

Now let one introduces the following ansatz for the expansion factor:

$$a(t) = e^{(H(t)t \,-\, H_0 T_0)/\beta} \tag{33}$$

where T_0 is the current age of the Universe, $H_0 = H(T_0)$ is the Hubble constant, and β is a constant parameter to be determined. From equations (21-23) one obtains

$$H(t) = H_0\left(\frac{t}{T_0}\right)^{\beta-1} \tag{34}$$

$$\dot{H}(t) = H(t)\frac{1}{t}\left(\beta - 1\right). \tag{35}$$

By inserting equations (34-35) into equation (25) one has:

$$\frac{\beta-1}{H_0 t}\left(\frac{t}{T_0}\right)^{1-\beta} + 1 = \frac{1}{\rho_{crit}}\left(-\frac{1}{2}\,\rho_m + \rho_\Lambda\right) \tag{36}$$

$$\frac{\beta-1}{H_0 T_0}\left(\frac{t}{T_0}\right)^{-\beta} = -\frac{1}{2}\,\Omega_m + \Omega_\Lambda - 1 \tag{37}$$

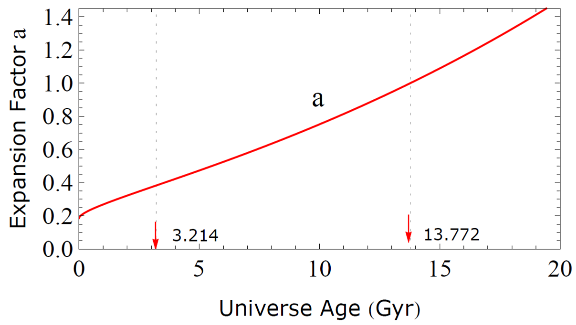

Fig. 1: $a(t) = e^{\frac{1}{\beta}\left(\left(\frac{t}{T_0}\right)^{\beta}-1\right)H_0 T_0}$

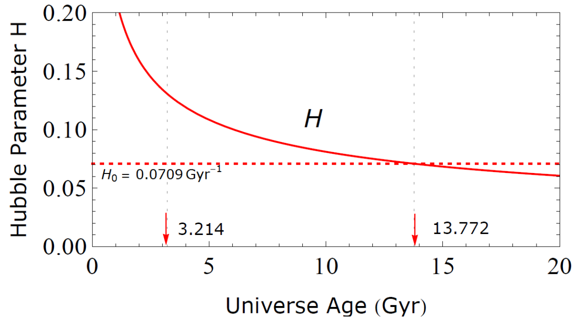

Fig. 2: $H(t) = H_0\left(\frac{t}{T_0}\right)^{\beta-1}$

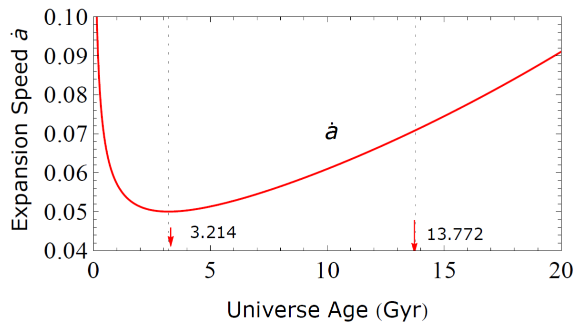

Fig. 3: $\dot{a}(t) = a(t)\, H_0\left(\frac{t}{T_0}\right)^{\beta-1}$

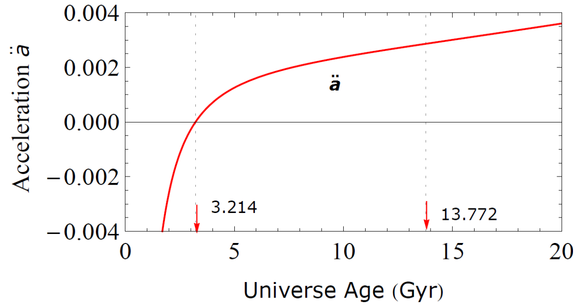

Fig. 4: $\ddot{a}(t) = a(t)\left(H_0\left(\frac{t}{T_0}\right)^{\beta} - (1-\beta)\,\frac{1}{t}\right)H_0\left(\frac{t}{T_0}\right)^{\beta-1}$

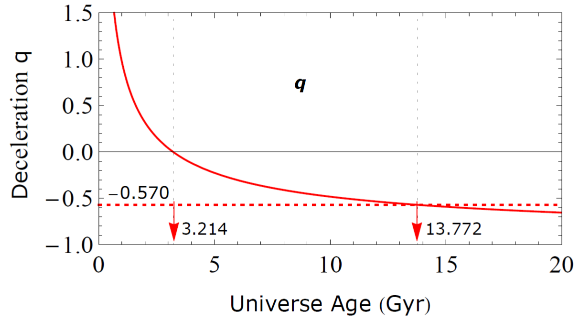

Fig. 5: $q(t) = -\left(1 + \frac{1}{H_0 T_0}\,(\beta-1)\left(\frac{t}{T_0}\right)^{-\beta}\right)$

Since β is assumed to be a constant, and, that $\Omega_m(T_0)$, $\Omega_\Lambda(T_0)$ and $H(T_0) = H_0$ are measured quantities, one has for $t = T_0$,

$$\frac{\beta-1}{H_0 T_0} = -\frac{1}{2}\,\Omega_m(T_0) + \Omega_\Lambda(T_0) - 1 \tag{38}$$

which solved for β gives

$$\beta = 1 + H_0 T_0\left(-\frac{1}{2}\Omega_m(T_0) + \Omega_\Lambda(T_0) - 1\right) = 0.5804. \tag{39}$$

where

$$H_0 = 69.32\ kms^{-1}Mpc^{-1} = 0.0709\ Gyr^{-1},$$

$$T_0 = 13.772\ Gyr,$$

$\Omega_m(T_0) = 0.2865$ and $\Omega_\Lambda(T_0) = 0.7135$ [6].

The plot of the expansion acceleration

$$\ddot{a}(t) = \left(\dot{H}(t) + H^2(t)\right)a(t) \tag{40}$$

as function of t = *age of the Universe* reveals that for $t < T_\star$, the acceleration is *negative* and for $t > T_\star$, the acceleration is *positive*. See Figure (4). This means that when the Universe is younger than $T_\star$, the regular gravitation overcomes dark energy, and after $T_\star$, dark energy overcomes gravitation. The result is an eternal positive accelerated expansion after $T_\star$ = 3.214 Gyr. See ahead.

In fact, by setting equation (40) to zero and just solving it for t,

$$H(t)\frac{1}{t}(\beta-1) + H^2(t) = 0, \tag{41}$$

one gets

$$t = T_\star = T_0\left(\frac{1-\beta}{H_0 T_0}\right)^{\frac{1}{\beta}} = 3.214\ Gyr\,. \tag{42}$$

From equation (26), one writes

$$\frac{c^2 k}{\mathfrak{R}^2(t) H^2(t)} = \Omega_m + \Omega_\Lambda - 1. \tag{43}$$

The known recently measured values of $\Omega_m(T_0)$ and $\Omega_\Lambda(T_0)$ [6] do not allow one to say, from above equation, that the

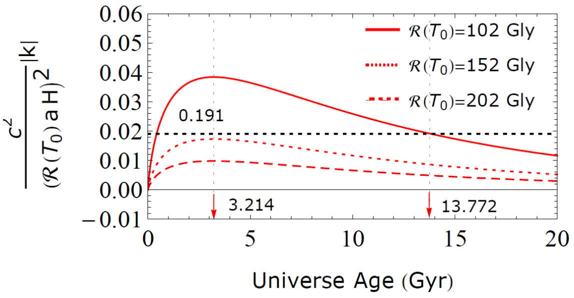

Fig. 6: Left hand side of equation (43) is plotted for some values of $\Re(T_0)$. At the current Universe age $T_0 = 13.772\,Gyr$, the right side of the referred equation has the margin of error equal to ± 0.0191.

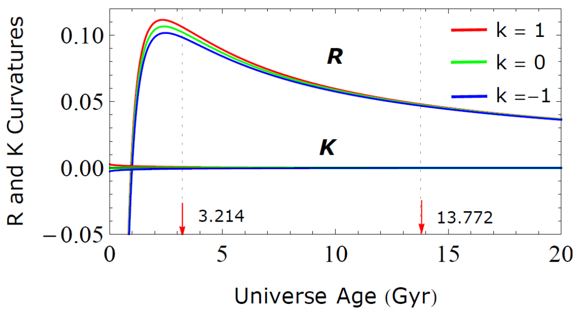

Fig. 7: Gaussian curvature $K(t) = \frac{k}{(\Re(T_0)a(t))^2}$ and Ricci scalar curvature $R(t) = 6\left(K(t) + \frac{1}{c^2}H(t)\left(2H(t) + \frac{1}{t}(\beta - 1)\right)\right)$.

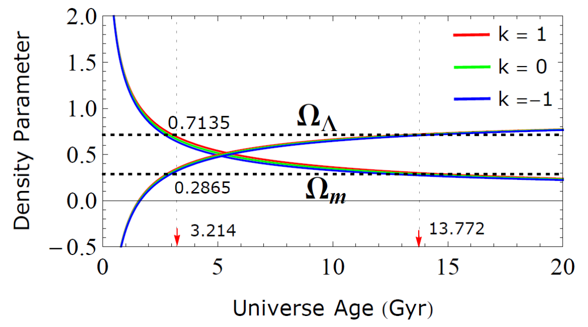

Fig. 8: Matter and dark energy density parameters for $k = 1, 0, -1$: $\Omega_m(t) = \frac{2}{3H^2(t)}\left(c^2K(t) - (\beta - 1)\frac{H(t)}{t}\right)$; $\Omega_\Lambda(t) = \frac{1}{3H^2(t)}\left(c^2K(t) + 2(\beta - 1)\frac{H(t)}{t} + 3H^2(t)\right)$. The radius of curvature is taken as $\Re(T_0) = 102\ Gly$.

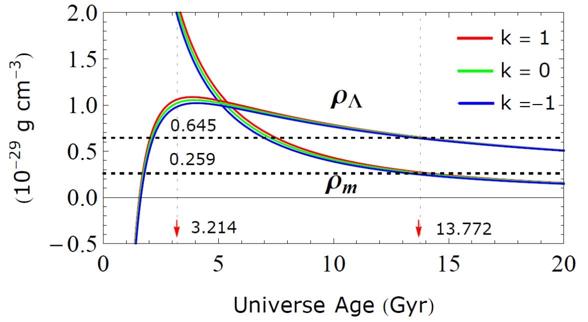

Fig. 9: Matter and dark energy densities for $k = 1, 0, -1$:
$\rho_m(t) = \frac{2}{8\pi G}\left(c^2K(t) - (\beta - 1)\frac{H(t)}{t}\right)$;
$\rho_\Lambda(t) = \frac{1}{8\pi G}\left(c^2K(t) + 2(\beta - 1)\frac{H(t)}{t} + 3H^2(t)\right)$.
The radius of curvature is taken as $\Re(T_0) = 102\ Gly$.

Universe is clearly flat ($k = 0$). The referred measured values have a margin of error:

$$\Omega_\Lambda(T_0) = 0.7135 \begin{cases} +0.0095 \\ -0.0096 \end{cases}$$

$$\Omega_m(T_0) = 0.2865 \begin{cases} +0.0096 \\ -0.0095 \end{cases}$$

Considering also the margin of errors of the other measured parameters [6], one cannot distinguish between $k = 1$, -1 or 0. The match between both sides of equations (43) requires that, the today's curvature radius of the Universe be $\Re(T_0) > 100\ Gly$, in the context of this paper. See Figure (6).

The so called deceleration parameter is

$$\begin{aligned} q(t) &= -\frac{\ddot{a}(t)a(t)}{\dot{a}^2(t)} = -\left(\frac{\dot{H}(t)}{H^2(t)} + 1\right) \\ &= -\left(1 + \frac{\beta - 1}{H_0T_0}\left(\frac{t}{T_0}\right)^{-\beta}\right) \end{aligned} \tag{44}$$

which, at current Universe age is $q(T_0) = -0.570$. See Figure (5).

The expansion scalar factor $a(t)$, Hubble parameter $H(t)$, expansion speed $\dot{a}(t)$, expansion acceleration $\ddot{a}(t)$, and the deceleration parameter $q(t)$ are plotted in Figures (1-5).

The sequence of Figures (7-10) shows the Gaussian $K(t)$ and R curvatures, matter density parameter $\Omega_m(t)$, dark energy density parameter $\Omega_\Lambda(t)$, matter density $\rho_m(t)$, dark energy density $\rho_\Lambda(t)$ and cosmological dark energy $\Lambda(t)$.

The time derivatives of $\rho_\Lambda(t)$ and $\rho_m(t)$ reveal interesting detail of the model in question:

$$\dot{\rho}_m + 3H\left(\rho_m + \frac{1}{c^2}p_m\right) = \dot{\rho}_m + 3H\rho_m = -Q \tag{45}$$

$$\dot{\rho}_\Lambda + 3H\left(\rho_\Lambda + \frac{1}{c^2}p_\Lambda\right) = \dot{\rho}_\Lambda = Q \tag{46}$$

$$Q = 2H\left(\frac{1}{t^2}(\beta - 2)(\beta - 1) + 3\dot{H} - c^2K\right) \tag{47}$$

where $p_m = 0$ and $p_\Lambda = -\rho_\Lambda c^2$. This implies that

$$\dot{\rho}_m + \dot{\rho}_\Lambda = -3H\rho_m. \tag{48}$$

The two perfect fluids interact with each other. In Figure (11) one shows the plots for $\dot{\rho}_m$, $\dot{\rho}_\Lambda$ and $\dot{\rho}_m + \dot{\rho}_\Lambda$ as functions of cosmic time.

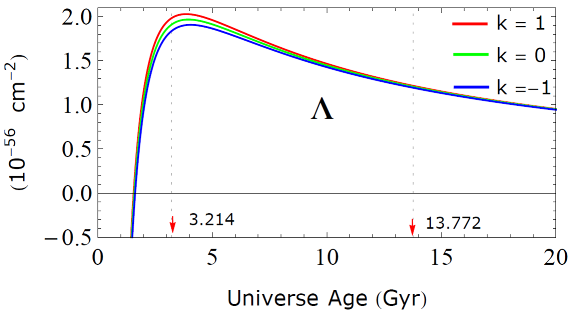

Fig. 10: Dark energy $\Lambda(t)$, in units of cm^{-2} for $k = 1, 0, -1$. $\Lambda(t) = \frac{1}{c^2} 8\pi G \rho_\Lambda(t)$. The radius of curvature is taken as $\Re(T_0) = 102\, Gly$. The result for $\Lambda(t)$ satisfies the following inequality: $|\Lambda| < 10^{-42} cm^{-2}$ [4].

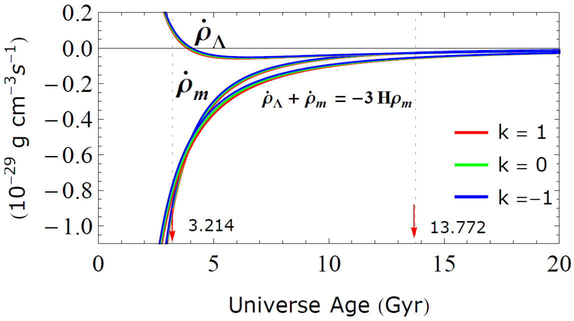

Fig. 11: Time derivatives of ρ_Λ, ρ_m and of the sum $\rho_\Lambda + \rho_m$ for $k = 1, 0, -1$. The radius of curvature is taken as $\Re(T_0) = 102\, Gly$.

4 Conclusion

The expression for the expansion factor $a(t) = e^{\frac{H_0 T_0}{\beta}\left(\left(\frac{t}{T_0}\right)^\beta - 1\right)}$, where $\beta = 0.5804$, constitutes a model for the expansion of the Universe for $t > 4 \times 10^{12} s \approx 1.3 \times 10^{-5} Gyr$ in which gravity dominates up to the Universe age of $T_* = 3.214\, Gyr$ and after that dark energy overcomes gravity and that persists forever. The acceleration of expansion is negative in the first part (*matter era*) and positive after that (*dark energy era*). The mathematical expressions for dark energy and matter densities indicate a clear interaction between the two perfect fluids (dark energy and matter). The classical deceleration parameter $q(t)$ is found to have the value $q(T_0) = -0.570$ for the current Universe age and the current radius of curvature should be $\Re(T_0) > 100\, Gly$.

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