Binding Energy and Equilibrium of Compact Objects

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The theoretical analysis of the existence of a limit mass for compact astronomic objects requires the solution of the Einstein's equations of general relativity together with an appropriate equation of state. Analytical solutions exist in some special cases like the spherically symmetric static object without energy sources that is here considered. Solutions, i.e. the spacetime metrics, can have a singular mathematical form (the so called Schwarzschild metric due to Hilbert) or a nonsingular form (original work of Schwarzschild). The former predicts a limit mass and, consequently, the existence of black holes above this limit. Here it is shown that, the original Schwarzschild metric permits compact objects, without mass limit, having reasonable values for central density and pressure. The lack of a limit mass is also demonstrated analytically just imposing reasonable conditions on the energy-matter density, of positivity and decreasing with radius. Finally the ratio between proper mass and total mass tends to 2 for high values of mass so that the binding energy reaches the limit m (total mass seen by a distant observer). As it is known the negative binding energy reduces the gravitational mass of the object; the limit of m for the binding energy provides a mechanism for stable equilibrium of any amount of mass to contrast the gravitational collapse.

1 Introduction to nonsingular Schwarzschild metric

The fate of extremely compact objects in the universe is ruled by the particular solutions of the Einstein's equations. As it is true that no all the mathematical theorems and statements have a corresponding meaning in the physical world, at the same time there is not a general rule, other than the verification by means of experimental and observational data, to establish, a priori, which mathematical solution must be discarded and which must be accepted.

In the case of the basic static model for compact objects, in the theory up to date the collapse is ruled by a specific solution (called Schwarzschild solution but not given explicitly by Schwarzschild, coming from the Hilbert's interpretation instead) that contains mathematical and thus physical singularities leading to a mass limit for ordinary compact objects and to the consequent black hole hypothesis (generalization to rotating or charged objects contains as well the features of singularity and horizon surface and it is not necessary in this context).

However, a different interpretation of the solution (nonsingular), particularly the original Schwarzschild solution, cannot be excluded if the completely different consequences (the nonexistence of mass limit and thus of black holes) are not yet demonstrated to be inconsistent with observational data.

1.1 Possible solutions to the static problem

Karl Schwarzschild in 1916 [1, eq. 14, page 194] gave an exact solution in vacuum to Einstein's field equation determining the line element for systems with static spherical symmetry (in units such that $c = G = 1$):

$$ds^2 = \left(1-\frac{\alpha}{R(r)}\right)dt^2 - \frac{dR(r)^2}{1-\dfrac{\alpha}{R(r)}} - R(r)^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \quad (1)$$

where α is a constant depending on the value of the mass, that can be obtained from the newtonian limit, and

$$R(r) = (r^3 + \sigma)^{1/3} \quad (2)$$

where σ (indicated with ρ in the original article) is a second constant to be determined and r is the same radial variable of the spherically symmetric Minkowski spacetime. Mathematically, there are two possible solutions that satisfy Einstein's field equation in vacuum ($R_{\mu\nu} = 0$): one is given by the class of infinite values of $R(r)$ such that [2, 3]

$$R(r) = (|r - r_0|^n + \alpha^n)^{1/n} \quad (3)$$

with arbitrary r_0 and $r \neq r_0$, the other is given by setting

$$R(r) = r. \quad (4)$$

It is worth to note that all the solutions of the class (3) can be obtained one from another by means of a simple coordinate transformation as must be in general relativity, while the solution (4) cannot be obtained from (3) and viceversa with a simple coordinate transformation. So, since the actual solution must be of course unique, the actual solution must be chosen among the form (3) and the form (4). At this stage, the only request that $R_{\mu\nu} = 0$ cannot discriminate about these solutions, additional considerations must be examined: in the following it will be shown that, since $R(r)$ is related to the Gaussian curvature, it cannot be set equal to the radial coordinate r as in (4) because this brings to unphysical consequences.

The choices made, for example, by Schwarzschild [1] ($r_0 = 0$, $r > r_0$, $n = 3$), by Brillouin [4] ($r_0 = 0$, $r > r_0$, $n = 1$) and by Droste [5] ($r_0 = \alpha$, $r > r_0$, $n = 1$) belong to the class of solutions of the first kind (3); all the solutions of this class share the same constant α in the denominator (or, like in the Droste's solution, the additional condition for validity that $r > \alpha$) that prevents the metric to become singular and to change signature so that they could be called a class of "nonsingular" solutions.

The other possibility is the "singular" solution (4), due to the contribution by Hilbert [6], leading to the so called "Schwarzschild Solution", that from now on will be called Schwarzschild-Hilbert or "singular" solution, that sets $n = 1$, $r_0 = \alpha$ in (3), so that $\sigma = 0$ in (2) i.e. $R = r$; this is similar to the Droste's solution but with no limitation on r so that $0 \leqslant r \leqslant \infty$. The line element in this case is the well known Schwarzschild (-Hilbert) metric

$$ds^2 = \left(1 - \frac{\alpha}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{\alpha}{r}} - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right), \quad (5)$$

where r is (supposed to be) the usual radial coordinates (but it is actually related to the Gaussian curvature as it will be shown later) running from zero to infinity and α is determined from the Newtonian potential in the limit $r \to \infty$, so that $\alpha = 2m$ where m is the mass in geometrized units while its complete expression would be $m = GM/c^2$.

The consequences of the line element (5) are well known, among them the existence of an "event horizon", a not removable singularity in $r = 0$, the change in the sign of the g_{00} and g_{11} elements of the metric when $0 \leqslant r \leqslant 2m$ and the existence of a mass limit for equilibrium of massive neutron cores [7] and the consequent black hole hypothesis.

There is an open question about if there is an actual difference between all these solutions, leading to different physical consequences. An example of this discussion can be find on references [2, 3, 8, 9].

The present article will not enter deep into the question, instead it must be intended as a contribute for understanding the possible physical consequences, on compact objects, applying the nonsingular metric (1 and 2).

1.2 Some characteristics of the Schwarzschild metric

This article, will start from a "nonsingular" solution, the one given by K. Schwarzschild [1] (1 and 2) (from now on, simply, Schwarzschild solution), that set (eq. 13 in [1])

$$\sigma = \alpha^3 = 8m^3 \quad (6)$$

so that the line element of the Schwarzschild Solution (1), using the coordinate r, becomes

$$ds^2 = \left(1 - \frac{\alpha}{(r^3+\sigma)^{1/3}}\right) dt^2 - \frac{r^4 (r^3+\sigma)^{-4/3}}{1 - \frac{\alpha}{(r^3+\sigma)^{1/3}}} dr^2 - (r^3+\sigma)^{2/3} \left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (7)$$

where σ has been explicitly left in order to compare all the subsequent formulas for this Schwarzschild metric (7) to the ones derived from the Schwarzschild-Hilbert metric (5), by simply setting $\sigma = 0$.

A first glance at the metric (7) indicates that there is no singularity at $r = 2m$, no "event horizon" and no change of sign (and of nature of the light cone) in the g_{00} and g_{11} elements of the metric. The "problem" has been moved to the origin $r = 0$ with the choice $\sigma = \alpha^3$. Moreover, the behavior of Schwarzschild metric, at the origin, is totally different from the one of Schwarzschild-Hilbert metric: in this latter, indeed, the presence of r in the denominator produces a mathematical, and consequently physical, not removable singularity, in the former there is just a smooth vanishing of the g_{00} and g_{11} metric elements, since in Schwarzschild metric (7)

$$\lim_{r \to 0} g_{00} = 0; \qquad \lim_{r \to 0} g_{11} = 0. \quad (8)$$

It worths to note that the expression of the "time" element g_{00} in the limit $r \to 0$ is analogous to the limit $r \to 2m$ of the same element in metric (5), so that there is a coordinate time (time measured by a distant observer) going to infinite while a radially ingoing object would approach $r = 0$.

Both singular (4) and nonsingular (3) class of solutions give similar results in the weak field limit, that is the limit where all the experimental proofs for general relativity are performed. For example, Schwarzschild, applied his metric (7) to solve the problem of the observed anomaly in the perihelion of Mercury. He found the exact solution ([1] eq.18 p.195) and noticed that the approximate Einstein's solution is the exact one by substituting the Einstein radial coordinate r with $(r^3+\alpha^3)^{1/3} = r(1+\alpha^3/r^3)^{1/3}$; since the term within parenthesis differs from 1 by a quantity of the order of 10^{-12}, the actual level of precision of the measurements cannot make a distinction between the two kind of metrics. Quite a different behavior appears in the strong field limit as it will be shown later.

1.3 Different nature of r and different centers of spherical symmetry for the two kind of metrics

The further analysis to discriminate among these two kind of metrics involves the nature of the r coordinate that represents two very different quantities in the two metrics. In effect can be demonstrated that, in the Schwarzschild metric (1), r is the usual radial coordinate analogue of the coordinate in

Minkowski space and $r = 0$ is the actual center of the configuration with a finite curvature: in the derivation of metric (1), Schwarzschild never changes the nature of r (see [1] eq.7) that corresponds to the radial coordinate of the Minkowski space. $r = 0$ corresponds to the center of the distribution and this is demonstrated if one looks at a curvature invariant, the Kretschmann scalar, that is maximized at $r = 0$ as it is required. In effect, considering the nonsingular Schwarzschild solution, its expression is

$$R_{kr} = R_{\mu\nu\lambda\xi} R^{\mu\nu\lambda\xi} = \frac{12\alpha^2}{\left(r^3 + \alpha^3\right)^2} \tag{9}$$

that has a maximum finite value in $r = 0$ of $R_{kr}(0) = 12/\alpha^4$.

At the same time, the Gaussian Curvature is defined by

$$K_S = \frac{R_{1212}}{g} = \frac{1}{R^2} = \frac{1}{(r^3 + \alpha^3)^{2/3}} \tag{10}$$

so that for $r = 0 \Rightarrow K_S = 1/\alpha^2$ so K_S is finite at the center.

On the other side, the r of the Schwarzschild-hilbert metric (5) it is not the radial coordinate neither a distance at all but it is, actually, the square of the inverse of the Gaussian curvature of a spherically symmetric geodesic surface in the spatial section of the spacetime manifold because

$$K_{SH} = \frac{R_{1212}}{g} = \frac{1}{r^2}\,. \tag{11}$$

Where are the centers of spherical distribution for the two kind of metric? The answer to this question can be given by the quantity that represents the *proper distance* $R_p(r) = \int g_{11} dr$.

In the Schwarzschild-Hilbert case (5),

$$\begin{aligned} R_p(r) &= \int g_{11} dr = \int \frac{1}{\sqrt{1 - \frac{\alpha}{r}}} dr = \\ &= \sqrt{r}\sqrt{r-\alpha} + \alpha \ln\left[2\left(\sqrt{r} + \sqrt{r-\alpha}\right)\right] + C \end{aligned} \tag{12}$$

where C is a constant. The center r_c of the distribution is found setting the proper distance equal to zero ($R_p(r_c) = 0$) that happens for $r_c = \alpha$ and $C = -\alpha \ln\left(2\sqrt{\alpha}\right)$. Finally the expression for the proper distance is [2, 3]

$$R_p(r) = \sqrt{r}\sqrt{r-\alpha} + \alpha \ln\left(\frac{\sqrt{r} + \sqrt{r-\alpha}}{\sqrt{\alpha}}\right). \tag{13}$$

So, in the Schwarzschild-Hilbert metric $\alpha \equiv 2m < r \leqslant \infty$, while the range of the proper distance is $0 \leqslant R_p \leqslant \infty$, there is no meaning for $r \leqslant 2m$ coherently with its nature connected with the Gaussian curvature and the center of the distribution is $r_c = 2m$.

This means that, if is given a Minkowski spacetime, where $\mathbf{E}^3$ is its Euclidean space, the center of the spherical symmetry is $r_c = 0$ and r coincides with the proper distance R_p and with the radius of Gaussian curvature R_G, $r = R_p = R_G$, considering the metric manifold $\mathbf{M}^3$, that is the spatial part of Schwarzschild-Hilbert spacetime, then the central point $R_p(r_c) = 0$ corresponds to the point $r_c = 2m$ in $\mathbf{E}^3$ that is any point on a spherical surface centered in $r = 0$ with radius $r = 2m$. Only in this way there is a one to one correspondence between all points of $\mathbf{E}^3$ and $\mathbf{M}^3$.

In the Schwarzschild case (7) instead,

$$\begin{aligned} R_p(r) &= \int g_{11} dr = \int \sqrt{\frac{r^4\left(r^3+\alpha^3\right)^{-\frac{4}{3}}}{1 - \frac{\alpha}{\left(r^3+\alpha^3\right)^{\frac{1}{3}}}}}\, dr = \\ &= \left(r^3+\alpha^3\right)^{-\frac{1}{3}} \times \sqrt{\left(r^3+\alpha^3\right)^{\frac{4}{3}} - \alpha\left(r^3+\alpha^3\right)} + \\ &+ \alpha \ln\left[2\left(r^3+\alpha^3\right)^{\frac{1}{6}} + 2\sqrt{\left(r^3+\alpha^3\right)^{\frac{1}{3}} - \alpha}\right] + C. \end{aligned} \tag{14}$$

The center of the distribution r_c if found setting $R_p(r_c) = 0$ that is for $r_c = 0$ and $C = -\alpha \ln\left(2\sqrt{\alpha}\right)$ so that the expression for the proper distance is

$$\begin{aligned} R_p(r) &= \left(r^3+\alpha^3\right)^{-\frac{1}{3}} \sqrt{\left(r^3+\alpha^3\right)^{\frac{4}{3}} - \alpha\left(r^3+\alpha^3\right)} + \\ &+ \alpha \ln\left[\frac{\left(r^3+\alpha^3\right)^{\frac{1}{6}} + \sqrt{\left(r^3+\alpha^3\right)^{\frac{1}{3}} - \alpha}}{\sqrt{\alpha}}\right]. \end{aligned} \tag{15}$$

In conclusion, in Schwarzschild metric (1) r is the actual radial coordinate that goes from 0 to ∞ (whole manifold) and $r = 0$ is recognized to be the center where the Kretschmann scalar is maximized (9) and the Gaussian Curvature $K_S(r) = 1/R(r)^2$ is finite since it goes from $K_S(0) = 1/\alpha^2$ to $K_S(\infty) = 0$. In Schwarzschild-Hilbert metric, (5) instead, r has nothing to do with the radial coordinate or distance but it is actually related to the Gaussian curvature $K_{SH} = 1/r^2$ and it is defined only from $2m$ to ∞ as recognized by Droste [5].

2 Metric inside matter and equilibrium equations

Let's consider a mass of degenerate matter (without source of energy [10]) in a finite volume, the full treatment consists in solving Einstein's equations (equilibrium equations) together with an appropriate equation of state for the matter. There are well known studies dedicated to the analysis of equilibrium in the strong field limit, for massive compact objects in the environment of the singular Schwarzschild-Hilbert metric, where neutron massive cores of neutron stars have been considered, imposing different equations of state for the neutron matter.

Anyway, all these different equations of state, from the pioneer and fundamental work of Oppenheimer and Volkoff [7] to the more realistic models [11] [12], share an important common characteristic: all these models, applied to the singular metric (5), predict some theoretical upper limit to a mass in equilibrium due to the intrinsic relativistic effect of the metric itself, and a consequent final collapse above this limit. The difference between these approaches regards the value of the limit that can change from 0.7 solar masses in the Oppenheimer-Volkoff (O-V) model to few solar masses in the other models [13]. Above these limits nothing can stop the object from the final collapse inside its "Schwarzschild" radius $2m$ and then, because of the changing of sign, up to a not avoidable final singularity, where curvature reaches an infinite value and the known physics meets its limits.

In this article, one of these models will be considered, in particular the O-V model in the environment of the nonsingular Schwarzschild metric (7) in the form valid inside the matter. The O-V model is not quite realistic because it considers the neutrons as a Fermi gas; however, no matter which model is considered, all the models predict a limit to the mass because of the singular metric, while it will be shown that in a nonsingular metric even the O-V model, that otherwise gives the sharper limit to the mass (≈ 0.7 of solar mass), does not show it, instead it gives the equilibrium radius for any value of the mass.

The procedure will follow the original one given by Oppenheimer and Volkoff so that the results can be directly compared. The difference will be that the nonsingular Schwarzschild metric inside matter will be applied instead of the singular one and the equations derived from the latter can be obtained from the former setting $\sigma = 0$.

Let's consider the static metric (7) with spherical symmetry, valid in empty space and set the g_{00} and g_{11} elements in the general exponential form:

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - \left(r^3+\sigma\right)^{2/3}\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (16)$$

Solving Einstein's equations (see Appendix A) the metric inside the matter is found:

$$ds^2 = \left(1 - \frac{2m(r)}{(r^3+8m^3)^{1/3}}\right)dt^2 - \frac{r^4\left(r^3+8m^3\right)^{-4/3}}{1-\dfrac{2m(r)}{(r^3+8m^3)^{1/3}}}dr^2 - \left(r^3+8m^3\right)^{2/3}\left(d\theta^2+\sin^2\theta d\phi^2\right). \quad (17)$$

The system of equilibrium equations becomes:

$$\left.\begin{aligned} \frac{dp(r)}{dr} &= -\frac{(p(r)+\varrho(r))\left[m(r)+4\pi\left(r^3+\sigma\right)p(r)\right]}{\dfrac{\left(r^3+\sigma\right)^{4/3}}{r^2}\left[1-\dfrac{2m(r)}{\left(r^3+\sigma\right)^{1/3}}\right]} \\ \frac{dm(r)}{dr} &= 4\pi\varrho(r)r^2 \end{aligned}\right\}. \quad (18)$$

where $\sigma = 8m^3$ and

$$m(r) = \frac{1}{2}\left(r^3+8m^3\right)^{1/3}\left(1 - e^{-\lambda}\frac{r^4}{(r^3+8m^3)^{4/3}}\right).$$

If one sets $\sigma = 0$ in the first equation of (18), then the Tolman-Oppenheimer-Volkoff equation (A-4) can be obtained; equations (18) together with an equation of state $\varrho = \varrho(p)$ constitute the system to be integrated.

3 Equation of state and numerical integration

Following the procedure by Oppenheimer and Volkoff [7], the matter is considered to consist of particles with rest mass μ_0 obeying Fermi statistics, neglecting thermal energy and forces between them; the equation of state can be put in the parametric form

$$\varrho = K\left(\sinh(t) - t\right),$$

$$p = \frac{1}{3}K\left(\sinh(t) - 8\sinh(t/2) + 3t\right),$$

where $K = \pi\mu_0^4 c^5 (4h^3$ and $t = 4\log(\hat{p}/\mu_0 c + [1+(\hat{p}/\mu_0 c)^2]^{1/2})$ where $\hat{p}$ is the maximum momentum in the Fermi distribution related to the proper particle density $N/V = 8\pi\hat{p}^3/(3h^3)$.

Setting $K = 1/4\pi$ the units of length a and of mass b are fixed such that, for neutron gas,

$$a = \frac{1}{\pi}\left(\frac{h}{\mu_0 c}\right)^{2/3}\frac{c}{(\mu_0 G)^{1/2}} = 1.36\times10^6\text{cm} \quad (19)$$

and $b = c^2 a/G = 1.83\times10^{34}$g.

Finally the system of adimensional equations, renaming the adimensional mass $m(r) \equiv u(r)$, to be integrated are

$$\left.\begin{aligned} \frac{du}{dr} &= r^2\left(\sinh(t)-t\right) \\ \frac{dt}{dr} &= -\frac{4(\sinh(t)-2\sinh(t/2))}{\dfrac{r^3+8m^3}{r^2}\left[\left(r^3+8m^3\right)^{1/3}-2u\right]}\times \\ &\quad\times\frac{\left[\frac{1}{3}\left(r^3+8m^3\right)(\sinh(t)+8\sinh(t/2)+3t)+u\right]}{\cosh(t)-4\cosh(t/2)+3} \end{aligned}\right\}. \quad (20)$$

This system is the analogous of the system integrated by Oppenheimer and Volkoff ([7], Eqs. 18 and 19) which can be obtained setting $\sigma \equiv \alpha^3 \equiv 8m^3 = 0$.

The procedure followed by Oppenheimer and Volkoff first fixes the value t_0 for the parameter t when $r = 0$ (determining central energy density and pressure), then the equations in [7] are numerically integrated for several finite values of t_0. Another boundary condition can be obtained setting of $u(0) \equiv u_0 = 0$. The equations are integrated till a value of $r = r_b$ for which t (and consequently the pressure) drops to 0, representing the border radius of the matter distribution; the corresponding value $u(r_b) = m$ is then, the value of the mass that can stay in equilibrium with a radius r_b and the imposed central density.

In the original paper (O-V) the first 4 results for t_0 equal to 1, 2, 3 and 4 are reported in a table (table I in [7], reported

Table 1: Comparison with Oppenheimer Volkoff table [7]; numbers not in parenthesis are in units a and b defined in (19).

	$m(M_s)$	$t_0(\varrho_0(10^{14}\text{g/cm}^3))$	r_b (km)
O-V	0.033 (0.30)	1.000 (1.014)	1.550 (21.1)
Eqs. (20)	0.033 (0.30)	1.006 (1.033)	1.506 (20.49)
O-V	0.066 (0.60)	2.000 (9.418)	0.980 (13.33)
Eqs. (20)	0.066 (0.60)	1.835 (6.923)	1.001 (13.61)
O-V	0.078 (0.71)	3.000 (40.62)	0.700 (9.52)
Eqs. (20)	0.078 (0.71)	2.166 (12.376)	0.861 (11.71)

here in table 1) together with an asymptotic value: the characteristics of the results is that, starting from $t_0 = 1$, the mass is increasing for increasing t_0 (the central density) but soon, for $t_0 = 3$, the mass reaches its maximum value calculated to be $M_{max} = 0.71$ solar masses.

Increasing further t_0, causes a decreasing of values for the mass (see [7], Fig. 1) so, for $m < M_{max}$ there are two values for central density but only the lower value must be considered to describe stable neutron stars; the maximum mass is thus considered the maximum possible mass for a stable equilibrium configuration of neutron stars with a Fermi equation of state as obtained by Oppenheimer and Volkoff. Different equations of state give different values of the maximum mass (till some units of solar masses) but anyway, as it will be seen later, a limit exists and is due to the use of the singular metric.

In our case, the equations to be integrated (20) came from the Schwarzschild nonsingular metric (17) so results can be quite different: in particular, there is an additional parameter that is the constant mass m, as seen by a distant observer. The integration procedure must then be modified: first, the parameter m is set and a prove of integration is performed starting from a low value of the central parameter t_0; integration on r ends at $r = r_b$, the border radius, where $t(r_b) = 0$ (null pressure): if the starting value t_0 is set too low, then the resulting mass would be $u(r_b) < m$. If this would be the case, then it would be necessary to increase t_0 to the minimum value such that $u(r_b) = m$. This minimum value t_0 together with m fixed and r_b found, will be the correct values for central density and pressure, mass and radius of the configuration in stable equilibrium.

For low values of the mass, i.e. for weak gravitational fields, results are expected to be similar to those of O-V while for increasing mass values the nonsingular metric should lead to results very different from those resulting from the singular one. In table I, the results are compared with the first three values of O-V table. It can be noted that for the lower mass (0.30 M_s), almost the same values are obtained for central density and radius, while on increasing the mass, the two approaches diverge and the nonsingular one leads to a "softer" equilibrium, with lower central density and greater radius, with respect to the O-V calculation.

If the mass is further increased, the two metrics behave in a complete different way: the O-V equations show a decreasing mass and a mass above the maximum found limit $0.71M_s$ cannot be sustained in equilibrium. On conversely, the nonsingular Schwarzschild metric will permit equilibrium for increasing masses and will not have a limit mass. The central density indeed will meet a maximum limit and, then, will decrease for increasing masses. At the same time the radius, instead of continuously decreasing for increasing masses as in O-V case, will show a minimum to keep the equilibrium configuration.

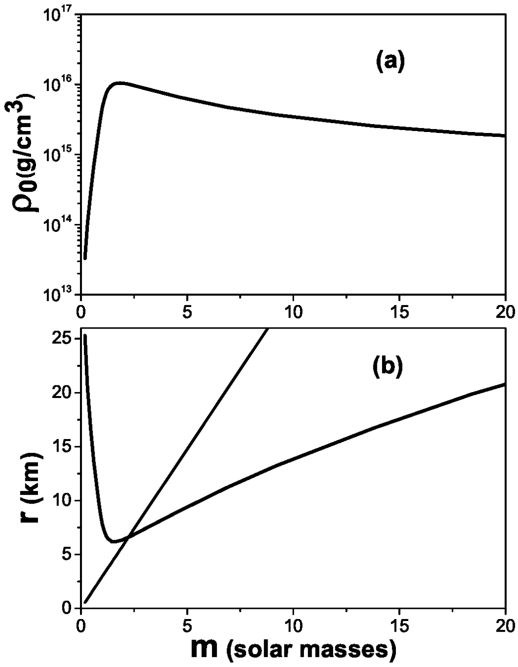

Fig. 1: Central density and equilibrium radius vs. mass: (a) central density shows a maximum; (b) equilibrium radius shows a minimum, straight line represents the so called Schwarzschild radius for that mass.

Let's first consider the behavior of various parameters for low masses: in Fig. 1 values of central density ρ_0 and radius r_b for low masses (up to 20 solar masses) are plotted; turning zones are clearly visible before the value of 2 solar masses in which the central density reaches a maximum and the radius a minimum. In particular, the central density reaches the maximum value of $1.048 \times 10^{16}\text{g cm}^{-3}$ at 1.84 solar masses while the radius reaches the minimum value of 6.172 km at 1.47 solar masses. It can be noted, in Fig. 1(b), that, in this zone, the equilibrium radius of the mass is below the value $r_b < 2m$ where $2m$ here is the constant in the denominator of the nonsingular metric and not a limit like the so called "Schwarzschild" radius for the singular metric.

The behavior of ρ_0 and r_b is, thus, totally different from the results obtained by Oppenheimer and Volkoff for the equilibrium with the singular metric; an interpretation for this behavior could derive from recalling the concept of proper mass

M^p, linked to the concept of gravitational binding energy E_B: the total mass m, i.e. the mass seen by a distant observer, is defined by $m = \int_0^{r_b} 4\pi\varrho(r)r^2 dr$ but if one integrates the energy-density ϱ over the proper "local" volume, the proper mass M^p of the system can be defined.

The proper volume element $d\tau$ is defined from $d\tau^2 = g_{ij}dx^i dx^j$ where $i, j = 1, 2, 3$ are only spatial coordinates. The proper volume from the O-V singular metric (5) then is $d\tau_S = 4\pi r^2(1 - 2m/r)^{-1/2}dr$ and the proper volume from the actual Schwarzschild nonsingular metric (7) $d\tau_N S = 4\pi r^2(1 - 2m/(r^3 + \sigma)^{1/3})^{-1/2}dr$; coherently can be defined respectively as two proper masses M^P:

$$M_S^P = \int_0^{r_b} \varrho 4\pi r^2(1 - 2m/r)^{-1/2}dr \tag{21}$$

and

$$M_{NS}^P = \int_0^{r_b} \varrho 4\pi r^2(1 - 2m/(r^3 + \sigma)^{1/3})^{-1/2}dr. \tag{22}$$

The physical meaning of proper mass is connected with the difference $M^P - m = E_B$ where E_B is the gravitational binding energy ([14] p. 126). In Fig. 2 the completely different behavior of the binding energy is shown, in the cases of singular solution and nonsingular solution: in the first case, the binding energy increases dramatically (together with the increasing of the central density to unphysical values) and above the maximum mass limit of about 0.7 solar masses the function becomes multivalued.

On the other side, in the nonsingular case, the binding energy increases smoothly with increasing mass and does not indicate any mass limit. In Fig. 2 only low mass values are reported but it will be shown later that, in the nonsingular case, the binding energy for higher mass values increases linearly with the mass and, considering that the ratio M^P/m in Fig. 3 tends $\to 2$, the binding energy tends to the value m of the rest mass.

Central (ϱ_0) and average $\varrho_{AV} \equiv M/(\frac{4}{3}\pi r_b^3)$ densities have a similar behavior: starting from values of $\varrho_0(0.184M_s) = 3.29 \times 10^{13} g/cm^3$ and $\varrho_{AV}(0.184M_s) = 5.40 \times 10^{12} g/cm^3$, reaching the maximum values of $\varrho_0(1.84M_s) = 1.0476 \times 10^{16} g/cm^3$ and $\varrho_{AV}(2.30M_s) = 3.688 \times 10^{15} g/cm^3$ and finally reaching the values for the last considered mass, $\varrho_0(3.68 \times 10^6 M_s) = 1.243 \times 10^{10} g/cm^3$ and $\varrho_{AV}(3.68 \times 10^6 M_s) = 8.687 \times 10^9 g/cm^3$.

Behavior evidences the presence of a maximum for both the densities and a decreasing for increasing masses: the central density converges to the average density values which decrease because volume grows with radius with an higher power than the mass.

Integration of the system (20) admits solution with an equilibrium radius for any amount of mass: in Fig. 3, higher values of mass are considered till, as an example, a value around 4 million of solar masses as it is supposed to be concentrated in the Milky Way's center.

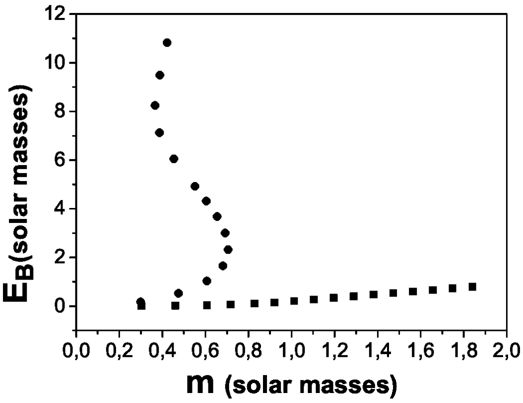

Fig. 2: Gravitational binding energy vs. mass: comparison between Oppenheimer-Volkoff results [7] (multivalued line with circles) and this article results (squares).

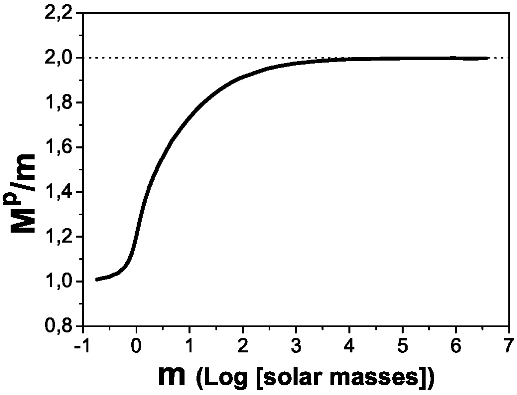

Fig. 3: Ratio between proper mass and mass vs. mass logarithm: limit tends to value 2 corresponding to an efficiency of 100% of mass conversion in gravitational binding energy

Together with the density decreasing with mass, there is another peculiar behavior, the one referred to the ratio of proper mass on mass: in Fig. 3 it is shown that this ratio tends to the value 2, meaning that there ia a 100% efficiency in converting mass into binding energy. The total mass of the compact object includes both the rest-mass energy and the negative binding energy so that the mass of the collapsed object is smaller than the sum of the component particles [15]. For neutron stars this mass deficit can be as large as 25% [16] but here it increases till 100% above 1 thousand of solar masses (depending on the equation of state) and this can be the mechanism to support stable equilibrium for such objects.

4 Inequality for nonexistance of a limit mass

Numerical results show that there is not a mass limit for equilibrium. This result can be seen also analytically trying to find an upper limit for the mass, independently from the specific equation of state. This limit exists in the case of singular metric and it is possible to see that this limit does not exist in the case of nonsingular metric following the procedure expressed, for example, by R.M. Wald [14, p. 130].

A first less sharp limit exists for the singular metric as necessary condition for the metric to be static: a metric is said to be static if it is stationary and, in addition, exists a spacelike hypersurface Σ (orthogonal to the timelike Killing vector field ξ^{α}); in order for Σ to be spacelike the necessary condition for staticity is that the radial element of the metric g_{11} would be greater than zero (in the following calculation, it will be used the Wald notation of $g_{11} \equiv h(r)$ and $g_{00} \equiv f(r)$, with the Suffix S to indicate the expression from the singular metric and NS for the nonsingular one).

So for the two metrics (5) and (7) it will be

$$h_S(r) = \left(1 - \frac{2m(r)}{r}\right)^{-1} \tag{23}$$

and

$$h_{NS}(r) = r^4 \left(r^3 + \sigma\right)^{-4/3} \left(1 - \frac{2m(r)}{(r^3 + \sigma)^{1/3}}\right)^{-1}. \tag{24}$$

The necessary condition for stability implies that, for a given mass M and equilbrium radius r_b, $h(r_b) > 0$ so, it clearly requires a limit for M only in the singular case, that is $M < r_b/2$ (eq. 6.2.32 in [14]) while, in the nonsingular case, $h_{NS}(r_b) > 0$ is always satisfied for any value of M and r_b (considering that $\sigma \equiv 8M^3$).

This limit for M (for the singular metric) can be sharpened using the condition $g_{00} \equiv f(r) \geqslant 0$ that imposes the Killing field ξ^{α} to be timelike everywhere. The term $f(r)$ has the form, for the singular and nonsingular metric, respectively

$$\left.\begin{aligned} f_S(r) &= \left(1 - \frac{2m(r)}{r}\right) \\ f_{NS}(r) &= \left(1 - \frac{2m(r)}{(r^3 + \sigma)^{1/3}}\right) \end{aligned}\right\}. \tag{25}$$

Since $f(r)$ must be greater than zero everywhere, it could seem that it would be necessary to know the specific equation of state for matter but, actually, the only conditions that must be assumed are very basic i.e. the density must be such that $\varrho \geqslant 0$ and $d\varrho/dr \leqslant 0$ while there is no need for whatsoever assumption about pressure P.

Applying these conditions, the following inequalities are obtained (see Appendix B): in the singular case it is found an upper mass limit

$$M \leqslant \frac{4}{9}\, r_b\,, \tag{26}$$

in the nonsingular case, instead, the following inequality is found:

$$1 - \left(\frac{8M^3}{r_b^3 + 8M^3}\right)^{\frac{1}{3}} \geqslant \frac{1}{9}\left(1 - \frac{8M^3}{r_b^3 + 8M^3}\right). \tag{27}$$

Since it is always true that $0 \leqslant 8M^3/(r_b^3 + 8M^3) \leqslant 1$, the inequality for the nonsingular case (27), i.e. the condition of stability, is always satisfied for any values of both M and r_b so that there is no upper limit for the mass, to have equilibrium, whatever would be the, reasonable, equation of state.

5 Conclusions

In conclusion, the application of the class of nonsingular static spherically symmetric metrics (particularly the Schwarzschild solution [1]) to the problem of hydrostatic equilibrium gives completely different solutions from those of the singular case. In this latter, there is a mass limit (whose value depends from the specific state equation) for dense cores of degenerate matter: above this limit, nothing can stop the configuration from a final gravitational collapse with formation of event horizon and inner physical singularity. In the case of nonsingular metric (that does not include the possibility of an event horizon) instead, the equilibrium is always reached whatever would be the amount of mass.

The application with a Fermi gas state equation, as in the Oppenheimer-Volkoff work [7], shows that central density has the same behavior, for increasing mass, than average density i.e. a maximum (with reasonable physical value), before reaching the 2 solar masses and then a decreasing. The equilibrium radius of the system shows a minimum before the 2 solar masses then grows with increasing masses but remaining well below the so called "Schwarzschild radius" for that mass which, in the nonsingular metric environment, is not the dimension of an event horizon but only a parameter connecting the general relativistic metric with the newtonian one. Proper mass of the system tends to the limit of twice the mass. This means that the negative binding energy tends to the limit of m counterbalancing the gravitational mass m. This is a mechanism that can stop gravitational collapsing and that can sustain stable equilibrium.

Considering experimental observations, weak field experiments give same results, within errors, for the singular and nonsingular metrics, while for strong fields, the nonsingular metric admits stable configuration of greater amount of mass while singular metrics admits black hole formation. Few observational, indirect, evidences for black holes existence have been performed in years but it seems that an alternative hypothesis of very compact degenerate matter configurations, permitted by nonsingular metrics, could be compatible with observations: let's consider, for example, a single nonrotating compact object of 9.2 solar masses (m=1 in units of (19)), in the singular metric, it would be a black hole, no matter of which state equation is used, and a "Schwarzschild radius" $r_s = 27.17$ km would define the horizon event whose surface would have an infinite gravitational redshift and would surround a pointlike singularity.

The application of nonsingular metric (with a Fermi equation of state) instead, would give a very compact object, of radius $r_b = 13.23$km, made by ordinary (degenerate) matter with a central density $\varrho_0 = 13.23 \times 10^{15} g/cm^3$; the density value is not far from the ordinary nuclear density, moreover a more realistic state equation would keep density value within reasonable physical limit.

Gravitational redshift factor $f = \sqrt{-g_{11}}$ (the ratio between wavelength observed at infinity and wavelength emitted at distance r) at the surface of the matter configuration would be $f = 1.165$. This redshift would correspond, in the black hole case, to a redshift of a photon emitted ad distance $r = r_s f^2/(f^2 - 1) = 3.8 r_s$. This difference, theoretically, could be observable but total luminosity would be so faint not to permit direct observations while indirect observations due to, for example, the accretion disk surrounding these compact objects, would be very similar. The existence of compact massive (several solar masses) objects could justify why observed emissions from individuated neutron stars and black hole candidates are so similar [17] despite the totally different characteristics of a hard surface and an event horizon.

Recent observations involving magnetic fields of quasars also put in doubt the existence of inner super-massive black holes [18]. It must be remarked that at the state of the art there is still no observational proof of a black hole event horizon [19].

Lack of single compact objects of very great mass it is due more to mechanism of formation of such object than to some mass limit, anyway in the galactic's centers there is gravitational evidence for compact objects of millions of solar masses. Let's resume how it would be such an object in the nonsingular model with a Fermi gas state equation (others EOS would not change the qualitative features): considering an object 3.6 millions of solar masses, it would have a radius of about $58,000$ km that is the half percent of its estimated "Schwarzschild radius" in the black hole hypothesis, a central density $\varrho_0 = 1.24 \times 10^{10} g/cm^3$ and a central pressure $P_0 = 7.3 \times 10^{16}$ Pa both smoothly decreasing outward.

Sagittarius A, the radio point source associated with the dark mass located at the center of the Milky Way, is the best studied black hole candidate to date, but till now has not be possible to verify or to exclude the presence of a horizon [20]. The horizon existence has been inferred because a surface emission, to remain undetected, would require large radiative efficiencies, greater than 99.6% [21] anyway, this is actually the phenomenon predicted by the application of nonsingular metric, because, as seen in Fig. 3, the limit value of 2 for the ratio M^P/M means an efficiency limit of about 100%. This could be justified, actually, by a not exotic object having a hard surface, emissions and gravitational effects compatible with observations, and that could be permitted because the contribution of the negative binding energy.

Submitted on February 16, 2014 / Accepted on February 24, 2014

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Appendix A

The only non vanishing components of the Einstein Tensor G are G_0^0, G_1^1 and $G_2^2 = G_3^3$. Considering a matter that supports no transverse stresses and has no mass motion then the energy momentum components are [22] $T_1^1 = T_2^2 = T_3^3 = -p$ and $T_0^0 = \varrho$ where p is the pressure and ϱ is the macroscopic energy density measured in proper coordinates. So Einstein's equations are

$$G_0^0 = 8\pi T_0^0 = 8\pi\varrho = e^{-\lambda}\left[\frac{\lambda' r^2}{r^3+\sigma} - \frac{r^4}{(r^3+\sigma)^2} - \right. \\ \left. -\frac{4r\sigma}{(r^3+\sigma)^2}\right] + \frac{1}{(r^3+\sigma)^{2/3}} \tag{T00}$$

$$G_1^1 = 8\pi T_1^1 = 8\pi p = e^{-\lambda}\left[\frac{\nu' r^2}{r^3+\sigma} + \frac{r^4}{(r^3+\sigma)^2}\right] - \\ -\frac{1}{(r^3+\sigma)^{2/3}} \tag{T11}$$

$$G_2^2 = 8\pi T_2^2 = e^{-\lambda}\left[\frac{(\nu'-\lambda')r^2}{2(r^3+\sigma)} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \right. \\ \left. + \frac{2r\sigma}{(r^3+\sigma)^2} + \frac{\nu''}{2}\right] + \frac{1}{(r^3+\sigma)^{2/3}} \tag{T22}$$

where p, ϱ, λ and ν are functions of r and the primes indicates a differentiation with respect to r. Since $T_1^1 = T_2^2$ then $(T_1^1 - T_2^2) \times 2/r = 0$ and from equations (T00) it is easy to verify that

$$\frac{d}{dr}\left(-T_1^1\right) + \left(T_0^0 - T_1^1\right)\frac{\nu'}{2} = \left(T_1^1 - T_2^2\right)\frac{2}{r} = 0 \tag{A-1}$$

so that this latter equation can be read

$$\frac{dp}{dr} = -\frac{p+\varrho}{2}\,\nu'. \tag{A-2}$$

Equations (T00), (T11) and (A-2) constitute the system of equations to be solved and correspond to the ones in Oppenheimer Volkoff article [7, Eqs. 4,3 and 5] if σ is set equal to 0; an opportune equation of state $\varrho = \varrho(p)$ must also be included in the system.

Eliminating ν' in (T11) and (A-2), the hydrostatic equilibrium equation in exponential form is

$$\frac{dp}{dr} = -\frac{p+\varrho}{2}\times \\ \times\left[8\pi p\, e^{\lambda}\frac{(r^3+\sigma)}{r^2} + e^{\lambda}\frac{(r^3+\sigma)^{1/3}}{r^2} - \frac{r^2}{(r^3+\sigma)}\right]. \tag{A-3}$$

If it is set $\sigma = 0$ and the singular metric (5) (inside the matter) is considered where $e^{\lambda(r)} = (1-2m(r)/r)^{-1}$ (and consequently $m(r) = \frac{1}{2}r(1-e^{-\lambda})$) then the Tolman-Oppenheimer-Volkoff equilibrium equation is obtained

$$\frac{dp}{dr} = -\frac{(p(r)+\varrho(r))\left[m(r)+4\pi r^3 p(r)\right]}{r^2\left(1-\dfrac{2m(r)}{r}\right)}. \tag{A-4}$$

In our case (A-3) instead, it is possible to give the correct physical meaning to $m(r)$ setting, for the nonsingular metric inside the matter,

$$e^{\lambda(r)} = \frac{(r^3+\sigma)^{-4/3}}{1-\dfrac{2m(r)}{(r^3+\sigma)^{1/3}}}\,r^4; \tag{A-5}$$

in effect, at the border $r = r_b$ there will be continuity with the metric in vacuum (7) and (6) so that

$$e^{\lambda(r_b)} = e^{\lambda} = \frac{r^4(r^3+8m^3)^{-4/3}}{1-\dfrac{2m}{(r^3+8m^3)^{1/3}}}$$

and $m(r_b)$ will assume its value m as seen by an external observer

$$m(r_b) = \frac{1}{2}\left(r_b^3+8m^3\right)^{1/3}\left(1-e^{-\lambda}\frac{r_b^4}{\left(r_b^3+8m^3\right)^{4/3}}\right) = m. \tag{A-6}$$

Finally the Schwarzschild metric inside the matter (in continuity with (7) where it is set $\alpha = 2m(r)$ and $\sigma = 8m^3$ so that $\sigma = \alpha^3$ outside the matter) will be

$$ds^2 = \left(1-\frac{2m(r)}{\left(r^3+8m^3\right)^{1/3}}\right)dt^2 - \\ -\frac{r^4\left(r^3+8m^3\right)^{-4/3}}{1-\dfrac{2m(r)}{\left(r^3+8m^3\right)^{1/3}}}\,dr^2 - \\ -\left(r^3+8m^3\right)^{2/3}\left(d\theta^2+\sin^2\theta d\phi^2\right). \tag{A-7}$$

So, with $e^{\lambda(r)}$ given by (A-5), the equilibrium equation (A-3) (that is the merging of the two Einstein's equations (T11) and (A-2)) and the other Einstein's equation (T00) will become respectively

$$\left.\begin{aligned} \frac{dp(r)}{dr} &= -\frac{(p(r)+\varrho(r))\left[m(r)+4\pi(r^3+\sigma)p(r)\right]}{\dfrac{\left(r^3+\sigma\right)^{4/3}}{r^2}\left[1-\dfrac{2m(r)}{\left(r^3+\sigma\right)^{1/3}}\right]} \\ \frac{dm(r)}{dr} &= 4\pi\varrho(r)r^2 \end{aligned}\right\}, \tag{A-8}$$

where $\sigma = 8m^3$.

Appendix B

Pressure P can be eliminated from Einstein's equations considering that $G^1_1 - G^2_2 = 0$, this, together with the definition of $h(r)$ (23) leads to the following equation for the singular metric (using the notation by Wald, eq. 6.2.34 in [14])

$$\frac{d}{dr}\left[r^{-1}h_S(r)^{-1/2}\frac{df_S^{1/2}(r)}{dr}\right] = \\ = [f_S(r)h_S(r)]^{1/2}\,\frac{d}{dr}\left[\frac{m(r)}{r^3}\right] \quad \text{(B-1)}$$

while, for the nonsingular metric

$$\frac{d}{dr}\left[(r^3+\sigma)^{-1/3}h_{NS}(r)^{-1/2}\frac{df_{NS}(r)^{1/2}}{dr}\right] = \\ = \frac{(r^3+\sigma)^{2/3}}{r^2}\,[f_{NS}(r)h_{NS}(r)]^{1/2}\,\frac{d}{dr}\left[\frac{m(r)}{r^3+\sigma}\right]. \quad \text{(B-2)}$$

The right sides for both equations are proportional to the derivative with respect to r of the average density, so because the condition $d\varrho/dr \leqslant 0$, the left sides must be both less or equivalent to 0. Integrating the inequalities for the left sides, inward from the border r_b to a generic radius r we obtain

$$\frac{1}{rh_S^{1/2}(r)}\frac{df_S(r)^{1/2}}{dr} \geqslant \frac{M}{r_b^3}\,, \quad \text{(B-3)}$$

$$\frac{1}{(r^3+\sigma)^{1/3}h_{NS}^{1/2}(r)}\frac{df_{NS}^{1/2}(r)}{dr} \geqslant \frac{M}{r_b^3+\sigma}\,. \quad \text{(B-4)}$$

These inequalities can be integrated again inward from r_b to 0. The condition $d\varrho/dr \leqslant 0$ implies that $m(r)$ cannot be smaller than the value it would have for a uniform density star so, for the singular case, $m(r) \geqslant Mr^3/r_b^3$ and, for the nonsingular one, $m(r) \geqslant M(r^3+\sigma)/(r_b^3+\sigma)$, so that inequalities (B-3 and B-4) become: for the singular case (Wald, eq. 6.2.39)

$$f_S^{1/2}(0) \leqslant \frac{3}{2}\left(1-\frac{2M}{r_b}\right)^{1/2} - \frac{1}{2} \quad \text{(B-5)}$$

and for the nonsingular case

$$f_{NS}^{1/2}(0) \leqslant \frac{3}{2}\left(1-\frac{2M}{(r_b^3+\sigma)^{1/3}}\right)^{1/2} - \frac{1}{2}\left(1-\frac{2M\sigma^{2/3}}{r_b^3+\sigma}\right) \quad \text{(B-6)}$$

(as usual for $\sigma = 0$ the two cases are equivalent). Finally, the condition $f^{1/2}(0) \geqslant 0$ implies that, for the singular case, the necessary condition for staticity involves a maximum limit for the mass: from (B-5)

$$M \leqslant \frac{4}{9}\,r_b\,. \quad \text{(B-7)}$$

For the nonsingular case instead, the stability condition implies, from (B-6) and inserting the value $\sigma \equiv 8m^3$, the inequality

$$1-\left(\frac{8M^3}{r_b^3+8M^3}\right)^{\frac{1}{3}} \geqslant \frac{1}{9}\left(1-\frac{8M^3}{r_b^3+8M^3}\right). \quad \text{(B-8)}$$