

Drude-Schwarzschild Metric and the Electrical Conductivity of Metals

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Starting from a string with a length equal to the electron mean free path and having a unit cell equal to the Compton length of the electron, we construct a Schwarzschild-like metric. We found that this metric has a surface horizon with radius equal to the electron mean free path and its Bekenstein-like entropy is proportional to the number of squared unit cells contained in this spherical surface. The Hawking temperature is inversely proportional to the perimeter of the maximum circle of this sphere. Also, interesting analogies on some features of the particle physics are examined.

1 Introduction

Drude model of the electrical conductivity of metals [1, 2], considers that in this medium the free electrons (the electrons in conductors) undergo Brownian motion with an average characteristic time τ between collisions. Due to the Pauli's exclusion principle, only the electrons with energies which are close to the Fermi energy participate in the conduction phenomena. These electrons travel freely on average by a distance called electron mean free path equal to $\ell = v_F \tau$, where v_F is the Fermi velocity.

Meanwhile, let us note the following feature of black hole physics [3]: an observer at a distance greater than R_S (the Schwarzschild or the surface horizon radius) of the black hole's origin, does not observe any process occurring inside the region bounded by this surface.

Going back to the phenomena of electrical conductivity in metals. let us consider (for instance in a copper crystal) an electron in the conduction band which just suffered a collision. In the absence of an external electric field, all the directions in space have equal probability to be chosen in a starting new free flight. Therefore if we take a sphere centered at the point where the electron have been scattered, with radius equal to the electron mean free path, the surface of this sphere may be considered as an event horizon for this process. Any electron starting from this center will be, on average, scattered when striking the event horizon, losing the memory of its previous free flight. Besides this, all lattice sites of the metallic crystal are treated on equal footing, due to the translational symmetry of the system.

This analogy between two branches of physics, general relativity (GR) and the electrical conduction in metals (ECM), will be considered in the present work. As we will see, we are going to use the GR tools to evaluate some basic quantities related to ECM. We are also going to use some concepts related to the study of particle lifetimes in particle physics (PP).

2 The electron mean free path as a Schwarzschild radius

Let us consider a string of length ℓ (coinciding with the electron mean free path), composed by N unit cells of size equal

to the Compton wavelength of the electron (λ_C). Associating a relativistic energy pc to each of these cells, we have an overall kinetic energy K given by

$$K = Npc = \frac{\ell}{\lambda_C} pc = \left(\frac{\ell mc^2}{h} \right) p. \quad (1)$$

In a paper entitled: "Is the universe a vacuum fluctuation?", E.P. Tryon [4] considers a universe created from nothing, where half of the mass-energy of a created particle just cancels its gravitational interaction with the rest of matter in the universe. Inspired by the Tryon proposal we can write

$$K + U = 0 \quad (2)$$

implying that

$$U = -K = - \left(\frac{\ell mc^2}{h} \right) p. \quad (3)$$

However, we seek for a potential energy which depends on the radial coordinate r , and by using the uncertainty relation $p = \frac{h}{r}$, we get

$$U = - \left(\frac{\ell mc^2}{r} \right). \quad (4)$$

Next we deduce a metric, in the curved space, which is governed by the potential energy defined in (4). We follow the procedure established in reference [5]. A form of equivalence principle was proposed by Derek Paul [6], and when it is applied to the potential energy (4) yields

$$\hbar d\omega = dU = \frac{\ell mc^2}{r^2} dr. \quad (5)$$

Now we consider de Broglie relation

$$\hbar\omega = 2mc^2. \quad (6)$$

Dividing (5) by (6) yields

$$\frac{d\omega}{\omega} = \frac{\ell}{2r^2} dr. \quad (7)$$

Performing the integration of (7) between the limits ω_0 and ω , and between R and r , we get

$$\omega = \omega_0 \exp\left(-\frac{\ell}{2}\left(\frac{1}{r} - \frac{1}{R}\right)\right) \quad (8)$$

and

$$\omega^2 = \omega_0^2 \exp\left(-\ell\left(\frac{1}{r} - \frac{1}{R}\right)\right). \quad (9)$$

Making the choice $R = \ell$, leads to

$$\omega^2 = \omega_0^2 \exp\left(1 - \frac{\ell}{r}\right) \quad (10)$$

Then we construct the auxiliary metric

$$d\sigma^2 = \omega^2 dt^2 - k^2 dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (11)$$

In (11) we take k^2 , such that

$$\frac{k^2}{k_0^2} = \frac{\omega_0^2}{\omega^2}. \quad (12)$$

Relation (12) is a reminiscence of the time dilation and space contraction of special relativity. Now we seek for a metric which becomes flat in the limit $r \rightarrow \infty$. This can be accomplished by defining [7]

$$\omega^2 = \ln\left(\frac{\omega^2}{\omega_0^2}\right), \quad \text{and} \quad k^2 = \frac{1}{\omega^2}. \quad (13)$$

Making the above choices we can write

$$ds^2 = \left(1 - \frac{\ell}{r}\right) dt^2 - \left(1 - \frac{\ell}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (14)$$

We observe that (14) is the Schwarzschild metric, where ℓ is just the Schwarzschild radius of the system.

3 A Schwarzschild-like metric

In the last section we deduced a metric where the so called Schwarzschild radius is just the conduction's electron mean free path. But that construction seems not to be totally satisfactory, once the viscous character of the fluid embedding the charge carriers has not yet been considered. By taking separately in account the effect of the viscous force, we can write

$$m \frac{dv}{dt} = -\frac{p}{\tau^*}. \quad (15)$$

In (15), τ^* is a second characteristic time, which differs from the first one τ that was defined in the previous section. Pursuing further we write

$$v dt = dr, \quad \text{and} \quad p = \frac{h}{r}. \quad (16)$$

Upon inserting (16) into (15), and multiplying (15) by v and integrating, we get the decreasing change in the kinetic energy of the conduction's electron as

$$\Delta K_{qt} = -\frac{h}{\tau^*} \ln\left(\frac{r}{R}\right), \quad (17)$$

where R is some radius of reference.

Next, by defining $\Delta U_{qt} = -\Delta K_{qt}$, we have the total potential energy U_t , namely

$$U_t = U + \Delta U_{qt} = -\frac{mc^2 \ell}{r} + \frac{h}{\tau^*} \ln\left(\frac{r}{R}\right). \quad (18)$$

In the next step, we consider the equivalence principle [6] and de Broglie frequency to a particle pair, writing

$$\frac{dU}{2mc^2} = \frac{d\omega}{\omega} = \frac{\ell}{2} \left(\frac{dr}{r^2}\right) + \frac{1}{2} \left(\frac{dr}{r}\right). \quad (19)$$

Upon integrating we get

$$\omega = \omega_0 \exp\left(-\frac{\ell}{2r} + \frac{1}{2} \ln\left(\frac{er}{\ell}\right)\right). \quad (20)$$

In obtaining (20), we have also made the choices

$$mc^2 \tau^* = h, \quad \text{and} \quad \frac{r}{R} = \frac{er}{\ell}. \quad (21)$$

Squaring (20), yields

$$\omega^2 = \omega_0^2 \exp\left(-\frac{\ell}{r} + \ln\left(\frac{er}{\ell}\right)\right). \quad (22)$$

Defining

$$\omega^2 = \ln\left(\frac{\omega^2}{\omega_0^2}\right), \quad \text{and} \quad k^2 = \frac{1}{\omega^2}, \quad (23)$$

we finally get

$$ds^2 = \left(\ln\left(\frac{er}{\ell}\right) - \frac{\ell}{r}\right) dt^2 - \left(\ln\left(\frac{er}{\ell}\right) - \frac{\ell}{r}\right)^{-1} dr^2 - r^2 d\Omega^2. \quad (24)$$

Relation (24) is a Schwarzschild-like metric [5], that displays the same qualitative behavior like that describing the Schwarzschild geometry. We also have used in (24) a compact form of writing the solid angle differential, namely $d\Omega$ (please compare with the last term of eq. (11)).

4 Average collision time as a particle lifetime

There are two characteristics linear momenta that we can associate to the free electrons responsible for the electrical conductivity of metals. They are the Fermi momentum mv_F and the Compton momentum mc . By taking into account the fermionic character of the electron, we will write a non-linear

Dirac-like equation describing the “motion” of this particle. We have [8]

$$\frac{\partial\psi}{\partial x} - \frac{1}{c} \frac{\partial\psi}{\partial t} = \frac{mv_F}{\hbar} \psi - \frac{mc}{\hbar} |\psi^* \psi|. \quad (25)$$

We see that eq. (25) contains only first order derivatives of the field ψ . Besides this, the field ψ exhibits not a spinorial character. Taking the zero of (25) and solving for $|\psi^* \psi|$, we get

$$|\psi^* \psi| = \frac{v_F}{c}. \quad (26)$$

On the other hand in the collision process, the conduction's electron loss its memory. We may think that this feature looks similar to the annihilation of a particle-anti particle pair, each of mass-energy equal to E_F . Putting this in a form of the uncertainty principle yields

$$2E_F \Delta t = \frac{h}{2} \quad \text{or} \quad \frac{h\nu}{2} = 2E_F. \quad (27)$$

Solving equation (27) for ν , we get

$$\nu = \frac{1}{\Delta t} = 4 \frac{E_F}{h}. \quad (28)$$

By combining the results of (28) and (26) we obtain the line width Γ tied to the “particle” decay

$$\Gamma = \nu |\psi^* \psi| = \frac{4E_F v_F}{hc}. \quad (29)$$

The averaged time between collisions τ is then given by

$$\tau = \frac{1}{\Gamma} = \frac{hc}{4E_F v_F}. \quad (30)$$

Now, let us compare the two characteristic times appearing in this work. By considering (21) and (30), we get

$$\frac{\tau}{\tau^*} = \frac{1}{2} \left(\frac{c}{v_F} \right)^3 \quad (31)$$

and the electron mean free path

$$\ell = v_F \tau = \frac{1}{2} \left(\frac{c}{v_F} \right)^2 \frac{h}{mc}. \quad (32)$$

Evaluating the number of unit cells in the string of size ℓ , we have

$$N = \frac{\ell}{\lambda_C} = \frac{mc^2}{4E_F}. \quad (33)$$

It is also possible to define an effective gravitational constant G_W as

$$\ell = 2 \frac{G_W N m}{c^2} = \frac{G_W m^2}{2E_F}. \quad (34)$$

Taking $M = Nm$, we can write

$$2 \frac{G_W M}{c^2} = \ell = \frac{G_W m}{v_F^2}, \quad (35)$$

which leads to

$$M = \frac{1}{2} m \left(\frac{c}{v_F} \right)^2. \quad (36)$$

In order to better numerically evaluate the quantities we have described in this work, let us take

$$E_F = \frac{1}{4} \alpha^2 m c^2. \quad (37)$$

This value for E_F [eq. (37)], is representative of the Fermi energy of metals, namely it is close to the Fermi energy of the copper crystal. Using (37) as a typical value of E_F , we get

$$\left(\frac{c}{v_F} \right)^2 = \frac{2}{\alpha^2}. \quad (38)$$

Inserting (38) into the respective quantities we want to evaluate, we have

$$\ell = \frac{h}{\alpha^2 m c}, \quad \tau = \frac{\sqrt{2} h}{\alpha^3 m c^2}, \quad M = \frac{m}{\alpha^2}. \quad (39)$$

Putting numbers in (39) yields

$$\ell = 453 \text{ \AA}, \quad \tau = 2.93 \cdot 10^{-14} \text{ s}, \quad M = 9590 \frac{\text{MeV}}{c^2}. \quad (40)$$

It would be worth to evaluate the strength of G_W . We have

$$G_W M^2 \sim 10^{-8} \hbar c. \quad (41)$$

We notice that M is approximately equal to ten times the proton mass.

5 The event horizon temperature and entropy

To obtain the Hawking [9, 11, 12] temperature of this model, we proceed following the same steps outlined in reference [5]. First, by setting $t \rightarrow i\tau$, we perform Wick rotation on the metric given by (24). We write

$$ds^2 = -(y d\tau^2 + y^{-1} dr^2 + r^2 d\Omega^2), \quad (42)$$

where y is given by

$$y = \ln \left(\frac{er}{\ell} \right) - \frac{\ell}{r}. \quad (43)$$

Now, let us make the approximation

$$y^{\frac{1}{2}} \sim \ell^{-\frac{1}{2}} \left(r \ln \left(\frac{er}{\ell} \right) - \ell \right)^{\frac{1}{2}} = \ell^{-\frac{1}{2}} u^{\frac{1}{2}}. \quad (44)$$

In the next step we make the change of coordinates

$$R d\alpha = \ell^{-\frac{1}{2}} u^{\frac{1}{2}} d\tau, \quad \text{and} \quad dR = \ell^{\frac{1}{2}} u^{-\frac{1}{2}} dr. \quad (45)$$

Upon integrating, taking the limits between 0 and 2π for α , from 0 to β for τ , and from ℓ to r for r , we get

$$R = \ell^{\frac{1}{2}} u^{\frac{1}{2}}, \quad \text{and} \quad R 2\pi = \ell^{-\frac{1}{2}} u^{\frac{1}{2}} \beta. \quad (46)$$

Finally from (46), we find the temperature T of the horizon of events, namely

$$T \equiv \frac{1}{\beta} = \frac{1}{2\pi\ell}. \quad (47)$$

Once we are talking about event's horizon, it would be worth to evaluate the Bekenstein [10–12] entropy of the model. Let us write

$$\Delta F = \Delta U - T\Delta S. \quad (48)$$

In (48), we have the variations of the free energy F , the internal energy U , and the entropy S . In an isothermal process, setting $\Delta F = 0$, and taking $\Delta U = Nmc^2$, and inserting T given by (47), we have

$$\Delta F = \left(\frac{\ell}{\lambda_C}\right)mc^2 - \frac{hc}{2\pi\ell}\Delta S = 0 \quad (49)$$

which leads to

$$\Delta S = 2\pi\left(\frac{\ell}{\lambda_C}\right)^2. \quad (50)$$

The entropy of the event's horizon is then (putting $S_0 = 0$)

$$S = S_0 + \Delta S = 2\pi\left(\frac{\ell}{\lambda_C}\right)^2. \quad (51)$$

6 Conclusion

Therefore the analogy developed in this work between black hole physics and the electrical conductivity of metals is very encouraging. This feature was discussed in a previous paper [8] where the connection with the cosmological constant problem [13] has also been considered

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