Weinberg Angle Derivation from Discrete Subgroups of SU(2) and All That

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The Weinberg angle θ_W of the Standard Model of leptons and quarks is derived from specific discrete (i.e., finite) subgroups of the electroweak local gauge group $SU(2)_L \times U(1)_Y$. In addition, the cancellation of the triangle anomaly is achieved even when there are four quark families and three lepton families!

1 Introduction

The weak mixing angle θ_W, or Weinberg angle, in the successful theory called the Standard Model (SM) of leptons and quarks is considered traditionally as an unfixed parameter of the Weinberg-Salam theory of the electroweak interaction. Its value of ∼30° is currently determined empirically.

I provide the only first principles derivation of the Weinberg angle as a further application of the discrete symmetry subgroups of SU(2) that I used for the first principles derivation of the mixing angles for the neutrino mixing matrix PMNS [1] in 2013 and of the CKM quark mixing matrix [2] in 2014. An important reminder here is that these derivations are all done within the realm of the SM and no alternative theoretical framework beyond the SM is required.

2 Brief review of neutrino mixing angle derivation

The electroweak component of the SM is based upon the local gauge group $SU(2)_L$ x $U(1)_Y$ acting on the two SU(2) weak isospin flavor states $\pm\frac{1}{2}$ in each lepton family and each quark family. Its chiral action, i.e., involving LH doublets and RH singlets, is dictated by the mathematics of quaternions acting on quaternions, verified by the empirically determined maximum parity violation. Consequently, instead of using SU(2) generators acting on SU(2) weak isospin states, one can equivalently use the group of unit quaternions defined by q = a + bi + cj + dk, for a, b, c, d real and $i^2 = j^2 = k^2 = ijk = -1$. The three familiar Pauli SU(2) generators σ_x, σ_y, σ_z, when multiplied by i, become the three generators k, j, i, respectively, for this unit quaternion group.

In a series of articles [3–5] I assigned three discrete (i.e., finite) quaternion subgroups (i.e., SU(2) subgroups), specifically 2T, 2O, 2I, to the three lepton families, one to each family (ν_e, e), (ν_μ, μ), (ν_τ, τ). These three groups permeate all areas of mathematics and have many alternative labelings, such as [3,3,2], [4,3,2], [5,3,2], respectively. Each of these three subgroups has three generators, $R_s = iU_s$ (s = 1,2,3), two of which match the two SU(2) generators, U_1 = j and U_3 = i, but the third generator U_2 for each subgroup is not k [6]. This difference between the third generators and k is the true source [1] of the neutrino mixing angles. All three families must act together to equal the third SU(2) generator k.

The three generators U_2 are given in Table 1, with $\phi = (\sqrt{5} + 1)/2$, the golden ratio. The three generators must add to make the generator k, so there are three equations for three unknown factors. The arccosines of these three normalized factors determine the quaternion angles 105.337°, 36.755°, and 122.459°. Quaternion angles are double angle rotations, so one uses their half-values for rotations in R^3, as assumed for the PMNS matrix. Then subtract one from the other to produce the three neutrino mixing angles $\theta_{12} = 34.29°$, $\theta_{23} = -42.85°$, and $\theta_{13} = -8.56°$. These calculated angles match their empirical values $\theta_{12} = \pm 34.47°$, $\theta_{23} = \pm (38.39°–45.81°)$, and $\theta_{13} = \pm 8.5°$ extremely well.

Table 1: Lepton Family Quaternion Generators U_2

Fam.	Grp.	Generator	Factor	Angle°
ν_e, e	332	$-\frac{1}{2}i - \frac{1}{2}j + \frac{1}{\sqrt{2}}k$	−0.2645	105.337
ν_μ, μ	432	$-\frac{1}{2}i - \frac{1}{\sqrt{2}}j + \frac{1}{2}k$	0.8012	36.755
ν_τ, τ	532	$-\frac{1}{2}i - \frac{\phi}{2}j + \frac{\phi^{-1}}{2}k$	−0.5367	122.459

Thus, the three mixing angles originate from the three U_2 generators acting together to become the k generator of SU(2). Note that I assume the charged lepton mixing matrix is the identity. Therefore, any discrepancy between these derived angles and the empirical angles could be an indication that the charged lepton mixing matrix has off-diagonal terms.

The quark mixing matrix CKM is worked out the same way [2] by using four discrete rotational groups in R^4, [3,3,3], [4,3,3], [3,4,3], [5,3,3], the [5,3,3] being equivalent to 2I × 2I. The mismatch of the third generators again requires the linear superposition of these four quark groups. The 3 × 3 CKM matrix is a submatrix of a 4 × 4 matrix. However, the mismatch of 3 lepton families to 4 quark families indicates a triangle anomaly problem resolved favorably in a later section by applying the results of this section.

3 Derivation of the Weinberg angle

The four electroweak generators of the SM local gauge group $SU(2)_L \times U(1)_Y$ are typically labeled W^+, W^0, W^-, and B^0, but they can be defined equivalently as the quaternion generators i, j, k and b. But we do not require the full SU(2) to act upon the flavor states $\pm\frac{1}{2}$ for discrete rotations in the unitary plane C^2 because the lepton and quark families represent specific discrete binary rotational symmetry subgroups of SU(2).

That is, we require just a discrete subgroup of $SU(2)_L \times U(1)_Y$. In fact, one might suspect that the 2I subgroup would be able to perform all the discrete symmetry rotations, but 2I omits some of the rotations in 2O. Instead, one finds that $2I \times 2I'$ works, where $2I'$ provides the "reciprocal" rotations, i.e., the third generator U_2 of 2I becomes the third generator U_2' for $2I'$ by interchanging ϕ and ϕ^{-1}:

$$U_2 = -\frac{1}{2}\,i - \frac{\phi}{2}\,j + \frac{\phi^{-1}}{2}\,k, \quad U_2' = -\frac{1}{2}\,i - \frac{\phi^{-1}}{2}\,j + \frac{\phi}{2}\,k. \quad (1)$$

Consider the three SU(2) generators i, j, k and their three simplest products: $i \times i = -1$, $j \times j = -1$, and $k \times k = -1$. Now compare the three corresponding $2I \times 2I'$ discrete generator products: $i \times i = -1$, $j \times j = -1$, and

$$U_2\; U_2' = -0.75 + 0.559i - 0.25j + 0.25k, \quad (2)$$

definitely not equal to -1. The reverse product $U_2'U_2$ just interchanges signs on the i, j, k, terms.

One needs to multiply this product quaternion U_2U_2' by

$$P = 0.75 + 0.559i - 0.25j + 0.25k \quad (3)$$

to make the result -1. Again, P′ has opposite signs for the i, j, k, terms only.

Given any unit quaternion $q = \cos\theta + \hat{n}\sin\theta$, its power can be written as $q^{\alpha} = \cos\alpha\theta + \hat{n}\sin\alpha\theta$. Consider P to be a squared quaternion $P = \cos 2\theta + \hat{n}\sin 2\theta$ because we have the product of two quaternions U_2 and U_2'. Therefore, the *quaternion* square root of P has $\cos\theta = \sqrt{0.75} = 0.866$, rotating the U_2 (and U_2') in the unitary plane C^2 by the quaternion angle of 30° so that each third generator becomes k. Thus the Weinberg angle, i.e., the weak mixing angle,

$$\theta_W = 30^\circ. \quad (4)$$

Therefore, the Weinberg angle derives from the mismatch of the third generator of $2I \times 2I'$ to the SU(2) third generator k.

The empirical value of θ_W ranges from 28.1° to 28.8°, values less than the predicted 30°. The reason for the discrepancy is unknown (but see [7]), although one can surmise either (1) that in determining the Weinberg angle from the empirical data perhaps some contributions have been left out, or (2) the calculated θ_W is its value at the Planck scale at which the internal symmetry space and spacetime could be discrete instead of continuous.

4 Anomaly cancellation

My introduction of a fourth quark family raises immediate suspicions regarding the cancellation of the triangle anomaly. The traditional cancellation procedure of matching each lepton family with a quark family "generation by generation" does produce the triangle anomaly cancellation by summing the appropriate $U(1)_Y$, $SU(2)_L$, and $SU(3)_C$ generators, producing the "generation" cancellation.

However, we now know that this "generation" conjecture is incorrect, because the derivation of the lepton and quark mixing matrices from the U_2 generators of the discrete binary subgroups of SU(2) above dictates that the 3 lepton families act as one collective lepton family for $SU(2)_L \times U(1)_Y$ and that the 4 quark families act as one collective quark family.

We have now created an effective single "generation" with one effective quark family matching one effective lepton family, so there is now the previously heralded "generation cancellation" of the triangle anomalies with the traditional summation of generator eigenvalues [8]. In the SU(3) representations the quark and antiquark contributions cancel. Therefore, there are no $SU(3) \times SU(3) \times U(1)$, $SU(2) \times SU(2) \times U(1)$, $U(1) \times U(1) \times U(1)$, or mixed U(1)-gravitational anomalies remaining.

There was always the suspicion that the traditional "generation" labeling was fortuitous because there was no specific reason for dictating the particular pairings of the lepton families to the quark families within the SM. Now, with the leptons and quarks representing the specific discrete binary rotation groups I have listed, a better understanding of how the families are related within the SM is possible.

5 Summary

The Weinberg angle derives ultimately from the third generator mismatch of specific discrete subgroups of SU(2) with the SU(2) quaternion generator k. The triangle anomaly cancellation occurs because 3 lepton families act collectively to cancel the contribution from 4 quark families acting collectively. Consequently, the SM may be an excellent approximation to the behavior of Nature down to the Planck scale.

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