Repulsive Gravity in the Oppenheimer-Snyder Collapsar

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The Oppenheimer-Snyder metric for a collapsing dust ball has a well defined equilibrium state when the time coordinate goes to plus infinity. The entire ball is contained within the gravitational radius r_0, but half of its content lies within a thin shell between r_0 and $0.94r_0$. This state has the acausal property that no light ray escapes from it, but if one boundary condition at the surface, which Oppenheimer and Snyder imposed without justification, is removed, then all points in the interior remain in causal contact by null geodesics with the exterior. This modification causes the half shell's interior radius to increase to $0.97r_0$. Together with the results of a previous article on the density inside a spherosymmetric neutron star, the present results indicate that, in contrast with the universal attraction of Newtonian gravity, General Relativity gives gravitational repulsion at high density.

1 Introduction

The modern concept of black hole originates with Chandrasekhar's [1] discovery of an upper bound for the mass of a Newtonian white dwarf; it has been claimed (see, for example [2] section 11.3) that the replacement of Newtonian gravitation by General Relativity (GR) makes no significant difference. Using GR, Oppenheimer and Volkoff [3] (OV) found a similar result for neutron stars, the upper bound being somewhat lower than in the white-dwarf case. The OV article, in its footnote 10, did indicate that the GR field equations allow for a stable solution having zero density at the origin in place of the maximum density there of the Newtonian solution, but gave no further attention to this possibility; there seems to have been no serious attempt to return to it since, though a well known text ([5] after equation 23.20) has described it as "unphysical". We showed [4] that solutions of the OV-footnote variety may easily be obtained. The only new feature of such solutions which could conceivably qualify for the "unphysical" label is that the metric has a simple-pole singularity at the origin. This singularity is curiously similar to that now very widely used to describe a black hole, but with the crucial difference that its residue is positive, so that instead of infinite density there we find zero density.

In our previous article we advocated a field, rather than the geometric interpretation of GR, constructing a field energy tensor to explain why the stellar material is concentrated in a spherical shell and not at the origin. Here we shall use an exclusively geometric description, but will nevetheless be able to demonstrate, by studying the particle geodesics inside the shell, that the picture which emerges almost demands that we accept there is gravitational repulsion in the interior of the shell. We conclude that the black hole is a Newtonian concept, superseded by GR.

Our geometric investigation is based on what seems to be the only time-dependent study of a collapsar, namely that of Oppenheimer and Snyder [6] (OS). In an early stage of black-hole theory this article's conclusion was seriously misquoted by Penrose [7] who stated:

"The general situation with regard to a spherically symmetrical body is well known [6]. For a sufficiently great mass, there is *no final equilibrium state* (our emphasis). When sufficient thermal energy has been radiated away, the body contracts and continues to contract until a physical singularity is encountered at $r = 0$."

OS did not say anything resembling this assertion of Penrose. Indeed we shall show below that the OS density distribution approaches a stationary distribution, whose diameter is twice the gravitational radius, as the time goes to plus infinity. It is true that OS also found that in this limit there is a region inside the collapsar from which light may not be emitted, but we shall show below that this is not a real property of the model, and that it may be easily repaired so that all points of the physical space, exterior and interior, remain causally connected at all times. Nobody has demonstrated that any real collapse situation leads to the "trapped surfaces" of the Penrose article, and I would argue that such surfaces would violate the kind of causality described in Weinberg's text ([2] section 7.5). This conclusion was also stated recently by Chafin [8].

2 The OS metric

OS used the comoving coordinates (τ, R, θ, ϕ) with the metric

$$ds^2 = d\tau^2 - \frac{8m^3R}{r}\,dR^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad (1)$$

$$r = 2m\left(R^{3/2} - \frac{3\tau}{4m}\right)^{2/3},$$

in the exterior region $R > 1$ and

$$ds^2 = d\tau^2 - \frac{r^2}{R^2}\,dR^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad (2)$$

$$r = 2mR\left(1 - \frac{3\tau}{4m}\right)^{2/3},$$

in the interior region $0 < R < 1$. By the transformation

$$t = \frac{4m}{3}R^{3/2} - \frac{2}{3}\sqrt{\frac{r^3}{2m}} - 2\sqrt{2mr} + 2m\ln\frac{\sqrt{r}+\sqrt{2m}}{\sqrt{r}-\sqrt{2m}}, \quad (3)$$

the exterior metric converts to the Schwarzschild form

$$ds^2 = \frac{r-2m}{r}dt^2 - \frac{r}{r-2m}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (4)$$

We note that, since R is a comoving coordinate, $R = const.$ is a freefall geodesic and in particular the surface $r_1(t)$, that is $R = 1$, satisfies

$$t = \frac{4m}{3} - \frac{2}{3}\sqrt{\frac{r_1^3}{2m}} - 2\sqrt{2mr_1} + 2m\ln\frac{\sqrt{r_1}+\sqrt{2m}}{\sqrt{r_1}-\sqrt{2m}}, \quad (5)$$

and also any such geodesic, for $R > 1$, has its speed v increasing up to a maximum $v = 2c/(3\sqrt{3})$ and then decreasing asymptotically to zero as r approaches $2m$. This confirms the OS statement [6] "…an external observer sees the star asymptotically shrinking to its gravitational radius".

For $0 < R < 1$, OS identified an "internal time" t by defining a *cotime* y as

$$t = \frac{4m}{3} - \frac{4m}{3}\sqrt{y^3} - 4m\sqrt{y} + 2m\ln\frac{\sqrt{y}+1}{\sqrt{y}-1}, \quad (6)$$

and then putting

$$y = \frac{r}{2mR} + \frac{R^2-1}{2}. \quad (7)$$

This not only gives a continuous match for the internal and external t at $R = 1$, but also the metric in $0 < R < 1$ is

$$ds^2 = \frac{2mr^2(y-1)^2}{Ry^3(r-2mR^3)}dt^2 - \frac{r}{r-2mR^3}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (8)$$

which is continuous with (4) at $R = 1$.

From (6) and (7) we now see that the equilibrium state of the OS model is given by

$$r = mR(3 - R^2) \quad (0 < R < 1), \quad (9)$$

which contradicts the conclusion stated by Penrose and quoted in the previous section of this article. The density ρ is obtained from the curvature tensor of (1)

$$\rho = \frac{mR^3}{4\pi r^3}, \quad (10)$$

and since

$$\sqrt{-g} = \frac{r^3}{R}\sin\theta, \quad (11)$$

it integrates over the volume of the collapsar to give

$$\int_{R<1}\rho\sqrt{-g}\,dR\,d\theta\,d\phi = m. \quad (12)$$

In the remote past, when $r \sim yR, y \to \infty$, the dust particles are distributed uniformly over the sphere's interior, but as collapse proceeds their trajectories, $R = const$, crowd near the surface. This may be shown by considering that in the remote past half of the particles are contained within a shell between $R = 2^{-1/3} = 0.7937$ and $R = 1$, and that their final positions are $r = mR(3 - R^2)$, so that they end up between $r = 1.881m$ and $r = 2m$.

3 A problem with causality

At no time does the entire content of the collapsar go inside the sphere $r = 2m$, so Figure 1 of Penrose [7] is an incorrect picture of the OS collapsar, as is the discussion about *trapped surfaces* on which the figure is based. There is, however a causal anomaly in the OS model, in that, for any $R < 1$, there is a value of t beyond which no light signal emerges.

For the region $R < 1$ we introduce the coordinates (x, R, θ, ϕ), where $x = r/(2mR)$. The metric is

$$\frac{1}{4m^2}ds^2 = xdx^2 - x^2dR^2 - x^2R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \quad (13)$$

A radial light wave or radial null geodesic (RNG) satisfies

$$\frac{dR}{dx} = -\frac{1}{\sqrt{x}}, \quad (14)$$

that is

$$R = 2\sqrt{x(0)} - 2\sqrt{x} = 2\sqrt{x_0} - 2\sqrt{x}. \quad (15)$$

In order to reach the surface at $R = 1$ we need $x(1) > 1$ and therefore $x_0 > 9/4$, but from (7) we find that the minimum value of x_0 is 3/2, reached at cotime $y = 1$, that is when t is plus infinity. It follows that, for $y > 7/4$, an RNG from the origin cannot escape.

It is a simple matter to repair this flaw in the OS model; we replace (7) by

$$y = 1 + x - \frac{(R-3)^2}{4}, \quad (16)$$

so that all RNGs for $y > 1$ escape and causality is preserved. I have established [9] that the metric tensor with these coordinates is again continuous at $R = 1$. It differs from (8) in that the tensor component g_{rt} is not zero in $R < 1$, but only at $R = 1$. It was the unjustified imposition of the condition $g_{rt} = 0$ which led OS to claim that the connection (7) between the (x, R) and the (t, r) coordinates is unique. Our amendment of the OS metric leads to a more concentrated shell, because the equilibrium state is now specified by

$$r = \frac{mR(3-R)^2}{2}, \quad (17)$$

which, putting $R = 2^{-1/3}$, leaves half of the original dust matter in a shell between $r = 1.932m$ and $r = 2m$.

4 Gravity becomes repulsive at high densities

We have all believed since 1687 that gravity is universally attractive, so it requires some effort to adjust to the idea that gravity may repel; even the new mode of thought which came with GR did not change the paradigm of attractive gravity. We have attempted to show elsewhere [4] how a full appreciation of the gravitational field may cause us to change our intuition. However, for the present article we shall stay within the geometric presentation of GR, merely pointing the way towards an understanding of repulsive gravity.

We consider the motion of a foreign dust particle of small mass which crashes radially into the surface $R = 1$ at time t, that is at the point $r = r_1(t)$ given by (5), with a speed greater than that at which the surface itself is moving. We ignore the gravitational force exerted by this foreign particle, so it moves along a radial geodesic of the metric (13). The coordinate R is cyclic, so we have a conservation equation

$$x^2 \frac{dR}{ds} = -C, \quad (C > 0), \tag{18}$$

and it then follows that

$$\frac{dR}{dx} = \frac{C}{\sqrt{C^2 x + x^3}}. \tag{19}$$

This equation, when integrated with initial conditions $(x, R) = (r_1/2m, 1)$, leads to a relation between the final values x_∞ and R_∞ at t equal to plus infinity

$$R_\infty = 1 - \int_{x_\infty}^{r_1/2m} \frac{C}{\sqrt{C^2 x + x^3}}\, dx. \tag{20}$$

Now substituting $y = 1$ in (16) provides a second such relation, so eliminating x_∞ we obtain R_∞ in terms of r_1 and C. This is not a difficult process numerically, but in the limiting ultrarelativistic case $C \to \infty$ – effectively a null geodesic – it becomes especially simple

$$R_\infty = 2 - \sqrt{r_1/(2m)}, \quad (r_1 < 8m). \tag{21}$$

If $r_1 > 8m$ such a particle passes through the centre and exits at the opposite end of the diameter. A particle which crashes into the collapsar when the latter is close to its final state – r_1 close to 1 – does not penetrate it beyond the surface shell described in the previous section.

As long as we stay within the constraints of the geometric interpretation of GR, we are not able to draw inferences about what causes such a dramatic deceleration; we could, for example [10], continue to insist that it results from time dilation of the metric. I suggest, however, that a return to the language of field theory offers us, at the very least, an attractive alternative; we may claim that the force of repulsive gravity which decelerates the incident particle is the very same as the one which compresses the particles of the collapsar into a thin shell. In the context of a collapsar having a more realistic equation of state we pursued this point of view in our previous article [4].

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