

Modified Standard Einstein's Field Equations and the Cosmological Constant

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The standard Einstein's field equations have been modified by introducing a general function that depends on Ricci's scalar without a prior assumption of the mathematical form of the function. By demanding that the covariant derivative of the energy-momentum tensor should vanish and with application of Bianchi's identity a first order ordinary differential equation in the Ricci scalar has emerged. A constant resulting from integrating the differential equation is interpreted as the cosmological constant introduced by Einstein. The form of the function on Ricci's scalar and the cosmological constant corresponds to the form of Einstein-Hilbert's Lagrangian appearing in the gravitational action. On the other hand, when energy-momentum is not conserved, a new modified field equations emerged, one type of these field equations are Rastall's gravity equations.

1 Introduction

In the early development of the general theory of relativity, Einstein proposed a tensor equation to mathematically describe the mutual interaction between matter-energy and spacetime as [13]

$$R_{ab} = \kappa T_{ab} \quad (1.1)$$

where κ is the Einstein constant, T_{ab} is the energy-momentum, and R_{ab} is the Ricci curvature tensor which represents geometry of the spacetime in presence of energy-momentum.

Einstein demanded that conservation of energy-momentum should be valid in the general theory of relativity since energy-momentum is a tensor quantity. This was represented as

$$T_{ab;b} = 0 \quad (1.2)$$

where semicolon (;) represents covariant derivatives. But equation (1.2) requires

$$R_{ab;b} = 0 \quad (1.3)$$

too which is not always true.

Finally, Einstein presented his standard field equations (EFEs) describing gravity in the tensor equations form, namely, [2–5, 8–12]

$$G_{ab} = \kappa T_{ab} \quad (1.4)$$

where G_{ab} is the Einstein tensor given by

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R \quad (1.5)$$

where, R , is the Ricci scalar curvature, and g_{ab} is the fundamental metric tensor.

In his search for analytical solution to his field equations he turned to cosmology and proposed a model of static and homogenous universe filled with matter. Because he believed of the static model for the Universe, he introduced a constant

term in his standard field equations to represent a kind of “anti gravity” to balance the effect of gravitational attractions of matter in it.

Einstein modified his standard equations by introducing a term to his standard field equations including a constant which is called the cosmological constant Λ , [7] to become

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab} \Lambda = \kappa T_{ab} \quad (1.6)$$

where Λ is the cosmological constant (assumed to have a very small value). Equation (1.6) may be written as

$$R_{ab} - \frac{1}{2} (R - 2\Lambda) g_{ab} = \kappa T_{ab} \quad (1.7)$$

Einstein rejected the cosmological constant for two reasons:

- The universe described by this theory was unstable.
- Observations by Edwin Hubble confirmed that the universe is expanding.

Recently, it has been believed that this cosmological constant might be one of the causes of the accelerated expansion of the Universe [15].

Einstein has never justified mathematically introduction of his cosmological constant in his field equations.

Based on that fact I have mathematically done that using simple mathematics.

2 Modified standard Einstein's field equations

I modified the (EFEs) by introducing a general function $L(R)$ of Ricci's scalar into the standard (EFEs). I do not assume a concrete form of the function. The modified (EFEs), then becomes

$$R_{ab} - g_{ab} L(R) = \kappa T_{ab} \quad (2.1)$$

Taking covariant derivative denoted by semicolon (;) of both sides of equation (2.1) yields

$$R_{ab;b} - [g_{ab}L(R)]_{;b} = \kappa T_{ab;b} \quad (2.2)$$

Since covariant divergence of the metric tensor vanishes, equation (2.2) may be written as

$$R_{ab;b} - g_{ab} \left(\frac{dL}{dR} \right) R_{;b} = \kappa T_{ab;b} \quad (2.3)$$

Substituting the Bianchi identity

$$R_{;c} = 2g^{ab}R_{ac;b} \quad (2.4)$$

in equation (2.3) and requiring the covariant divergence of the energy-momentum tensor to vanish (i.e. energy-momentum is conserved), namely, equation (1.2), we arrive at

$$R_{ab;b} - g_{ab} \left(\frac{dL}{dR} \right) (2g^{ac}R_{ab;c}) = 0 \quad (2.5)$$

Rearranging equation (2.5) we get

$$R_{ab;b} - 2 \left(\frac{dL}{dR} \right) (g_{ab}g^{ac}) R_{ab;c} = 0 \quad (2.6)$$

Substituting the following identity equation

$$g_{ab}g^{ac} = \delta_b^c \quad (2.7)$$

in equation (2.6), we get

$$R_{ab;b} - 2 \left(\frac{dL}{dR} \right) (\delta_b^c) R_{ab;c} = 0 \quad (2.8)$$

By changing the dummy indices, we arrive at

$$R_{ab;b} \left(1 - 2 \frac{dL}{dR} \right) = 0 \quad (2.9)$$

We have either,

$$R_{ab;b} = 0, \quad (2.10)$$

or

$$1 - 2 \left(\frac{dL}{dR} \right) = 0 \quad (2.11)$$

Equation (2.10) is not always satisfied as mentioned before. Whilst, equation (2.11) yields

$$\frac{dL}{dR} = \frac{1}{2} \quad (2.12)$$

This has a solution

$$L(R) = \frac{1}{2} R - C \quad (2.13)$$

where C is a constant.

Interpreting the constant of integration C , as the cosmological constant Λ , the functional dependence of $L(R)$ on Ricci scalar may be written as

$$L(R) = \frac{1}{2} (R - 2\Lambda) \quad (2.14)$$

Equation (2.14) is the well known Lagrangian functional of the Einstein-Hilbert action with the cosmological constant.

3 The Modified Equations and the Einstein Spaces

In absence of energy-momentum i.e. in a region of spacetime where there is no energy, a state which is different from vacuum state everywhere in spacetime, equation (2.1) becomes

$$R_{ab} - g_{ab}L(R) = 0 \quad (3.1)$$

Contacting equation (3.1) with g^{ab} , we get

$$R - NL(R) = 0 \quad (3.2)$$

where N is the dimension of the spacetime. Equation (3.2) yields

$$L(R) = \frac{1}{N} R \quad (3.3)$$

Substituting equation (3.3) in equation (3.1), we get

$$R_{ab} = \frac{1}{N} g_{ab}R \quad (3.4)$$

Equation (3.4) is the Einstein equation for Einstein spaces in differential geometry [1, 2];

$$R_{ab} = I g_{ab} \quad (3.5)$$

where I is an invariant. This implies that the function I proposed, $L(R)$, is exactly the same as the invariant I in Einstein spaces equation when contacted with g^{ab} .

A 2D sections of the 4D spacetime of Einstein spaces are geometrically one of the geometries of spacetime which satisfies the standard Einstein's field equations in absence of energy-momentum.

A naive substitution of $N = 4$ into equation (3.4) would lead to an identity from which Ricci scalar could not be calculated, because it becomes a non-useful equation, it gives $R = R$.

4 The modified equations and gravity equations with non-conserved energy-momentum

Because in general relativity spacetime itself is changing, the energy is not conserved, because it can give energy to the particles and absorb it from them [2].

In cosmology the notion of dark energy – represented by term introduced by Einstein – and dark matter is a sort of sources of energy of unknown origin.

It is possible to incorporate the possibility of non-conserved energy-momentum tensor in the modified equations. In this case equation (2.9) should become

$$R_{ab;b} \left(1 - 2 \frac{dL}{dR} \right) = \kappa T_{ab;b} \quad (4.1)$$

where $T_{ab;b} \neq 0$. Since $R_{ab;b}$ is not always equals to zero, this implies that the bracket in the LHS of equation (4.1) is not zero in any case.

Let us assume it is equal to D , where D is a dimensionless constant, i.e.

$$1 - 2\frac{dL}{dR} = D \quad (4.2)$$

Then, equation (4.2) becomes

$$\frac{dL}{dR} = \frac{1}{2}(1 - D) \quad (4.3)$$

Now, integrating equation (4.3) yields

$$L(R, D) = \frac{1}{2}(1 - D)R - E \quad (4.4)$$

where E is a constant. When $D = 0$, equation (4.4) should reduce to equation (2.13), the equation in case of conserved energy-momentum, for which $E = \Lambda$. So, equation (4.4) becomes

$$L(R, D) = \frac{1}{2}(1 - D)R - \Lambda \quad (4.5)$$

Finally, the modified equations (equation (2.1)) in case of non-conserved energy-momentum become

$$R_{ab} - \frac{1}{2}(1 - D)g_{ab}R + \Lambda g_{ab} = \kappa T_{ab} \quad (4.6)$$

5 The modified equations and the Rastall gravity equations

Rastall [14] introduced a modification to the Einstein field equations, in which the covariant conservation condition $R_{ab;b} = 0$ is no longer valid.

In his theory he introduced a modification to the Einstein field equations without the cosmological constant which read

$$R_{ab} - \frac{1}{2}(1 - 2\lambda\kappa)g_{ab}R = \kappa T_{ab} \quad (5.1)$$

where λ is a free parameter. When $\lambda = 0$, we recover the standard Einstein's field equations. Comparing Rastall's equations in equation (5.1) with equation (4.6) without the cosmological constant, we deduce

$$D = 2\lambda\kappa \quad (5.2)$$

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