Instanton Representation of Plebanski Gravity. Gravitational Instantons from the Classical Formalism

Eyo Eyo Ita III*

Abstract: We present a reformulation of General Relativity as a "generalized" Yang-Mills theory of gravity, using a SO(3,C) gauge connection and the self-dual Weyl tensor as dynamical variables. This formulation uses Plebanski's theory as the starting point, and obtains a new action called the instanton representation of Plebanski gravity (IRPG). The IRPG has yielded a collection of various new results, which show that it is a new approach to General Relativity intrinsically different from existing approaches. Additionally, the IRPG appears to provide a realization of the relation amongst General Relativity, Yang-Mills theory and instantons.

Contents:

*Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom. E-mails: eei20@cam.ac.uk; ita@usna.edu.

§1. Introduction. In the 1980's there was a major development in General Relativity due to Abhay Ashtekar, which provided a new set of Yang-Mills like variables known as the Ashtekar variables (see e.g. [1, 2] and [3]). These variables have re-invigorated the efforts at achieving a quantum theory of gravity using techniques from Yang-Mills theory. Additionally, the relation of General Relativity to Yang-Mills theory by its own right is an interesting and active area of research [4,5]. The purpose of the present paper is two-fold. First, we will provide a new formulation of General Relativity which shows that its relation to Yang-Mills theory can be taken more literally in a certain well-defined context. The degrees of freedom of General Relativity will be explicitly embedded in a Yang-Mills like action resembling an instanton, and this formulation will be referred to as the instanton representation of Plebanski gravity. Secondly, in this paper we will focus just on some of the classical aspects of the theory, and make contact with existing results of General Relativity as well as provide various new results.

The organization of this paper is as follows. In §2 we will first provide a review of Plebanski's theory of gravity I_{Pleb} and the mechanism by which the Einstein equations follow from it. The Plebanski action contains a self-dual connection one-form A^a, where $a = 1, 2, 3$ denotes an SO(3,C) index with respect to which the (internal) self-duality is defined, a matrix $\psi_{ae} \in$ SO(3,C) $\otimes$ SO(3,C), and a triple of self-dual two-forms Σ^a, also self-dual in the SO(3,C) sense. The Ashtekar action I_{Ash} arises upon elimination of ψ_{ae} via a new mechanism, which basically restricts one to a functional submanifold of the space of actions defined by I_{Pleb}. Using this same mechanism, in §3 we show that elimination of certain components of the two forms Σ^a in favor of ψ_{ae} yields a new action I_{Inst}, the instanton representation of Plebanski gravity. This shows that I_{Ash} and I_{Inst} are in a sense complementary within I_{Pleb}, which suggests that the latter is also a theory of General Relativity. We prove this rigorously in §4 by demonstrating that I_{Inst} does indeed reproduce the Einstein equations, combined with a prescription for writing a solution subject to the initial value constraints.

In §5 we provide an analysis of the I_{Inst} equations of motion beyond the Einstein equations. A Hodge duality condition emerges on-shell, which as shown in §7 explicitly provides the spacetime metric.* In §6 we clarify the similarities and differences between I_{Inst} and the pure spin

*The implication is that the metrics from §4 and §7 must be equal to each other as a consistency condition. This should provide a practical method for constructing General Relativity solutions via what we will refer to as the instanton representation method.

connection formulation of Capovilla, Dell and Jacobson (CDJ) in [6]. There are common notions in the community that a certain antecedent of the CDJ action is essentially the same action as I_{Inst}. The present paper shows that I_{Inst} is in fact a new action for General Relativity. This will as well be independently corroborated by various follow-on papers which apply the instanton representation method to the construction of solutions. §7 delineates the reality conditions on I_{Inst}, which appear to be intertwined with the signature of spacetime. §8 and §9 clarify a hidden relation of General Relativity to Yang-Mills theory, which brings into play the concept of gravitational instantons.

The author has not been able to find, amongst the various sources in the literature, a uniform definition of what a gravitational instanton is. Some references, for example as in [7] and [8], define gravitational instantons as General Relativity solutions having a vanishing Weyl tensor with nonvanishing cosmological constant. This would seem to imply, in the language of the present paper, that gravitational instantons can exist only for spacetimes of Petrov Type O.* On the other hand, other references (for example [9]) allow for Type D gravitational instantons. In spite of this a common element, barring topological considerations, appears to be that of a solution to the vacuum Einstein equations having self-dual curvature. We hope in the present paper to shed some light on the concept of gravity as a "generalized" Yang-Mills instanton, which can exist as a minimum for Petrov Type I in addition to Types D and O. §10 contains a summary of the main results of this paper and some future directions of research, touching briefly on the quantum theory.

On a final note prior to proceeding, we will establish the following index conventions for this paper. Lowercase symbols from the beginning part of Latin alphabet $a, b, c, \dots$ will denote internal SO(3,C) indices and those from the middle $i, j, k, \dots$ will denote spatial indices, each taking values 1, 2 and 3. SL(2,C) indices will be labelled by capital letters A and A' taking values 0 and 1, and four-dimensional spacetime indices by Greek symbols $\mu, \nu, \dots$. For the internal SO(3,C) indices $(a, b, c, \dots h)$ the raised and lowered index positions are equivalent since the SO(3) group metric is taken to be the unit matrix (e.g. $\delta_{ab} \equiv \delta^a_b \equiv \equiv \delta^b_a \equiv \delta^{ab}$). For spatial indices $(i, j, k, \dots)$ and spacetime indices $(\mu, \nu, \dots)$, the raised and the lowered index positions are not equivalent, since the corresponding covariant metrics h_{ij} and $g_{\mu\nu}$ are in general different from the unit matrix. For multi-indexed quantities we will

*The definitions of the various Petrov Types can be found in [10] and in [11]. The purpose of the instanton representation of Plebanski gravity is to be able to classify General Relativity solutions according to their Petrov Type.

normally separate SO(3,C) from the other types of indices by placing them in opposing positions. So for example, the objects A^a_i and B^i_a respectively will be used to denote a SO(3,C) gauge connection and its associated magnetic field.

§2. Plebanski's theory of gravity. The starting Plebanski action [12] writes General Relativity using self-dual two forms in lieu of the spacetime metric $g_{\mu\nu}$ as the basic variables. We adapt the starting action to the language of the SO(3,C) gauge algebra as

$$I = \int_{\mathrm{M}} \delta_{ae}\,\Sigma^a \wedge F^e - \frac{1}{2}\big(\delta_{ae}\,\varphi + \psi_{ae}\big)\Sigma^a \wedge \Sigma^e, \tag{1}$$

where $\Sigma^a = \frac{1}{2}\Sigma^a_{\mu\nu}\,dx^\mu \wedge dx^\nu$ are a triplet of SO(3,C) two forms and $F^a = \frac{1}{2}F^a_{\mu\nu}\,dx^\mu \wedge dx^\nu$ is the field-strength two form for gauge connection one form $A^a = A^a_\mu dx^\mu$. Also, ψ_{ae} is symmetric and traceless and φ is a numerical constant. The field strength is written in component form as $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu$, with SO(3,C) structure constants $f^{abc} = \epsilon^{abc}$. The equations of motion resulting from (1) are (see e.g. [13] and [14])

$$\left.\begin{aligned} &\frac{\delta I}{\delta A^g} = \mathrm{D}\Sigma^g = d\Sigma^g + \epsilon^g_{fh}A^f \wedge \Sigma^h = 0\\ &\frac{\delta I}{\delta \psi_{ae}} = \Sigma^a \wedge \Sigma^e - \frac{1}{3}\,\delta^{ae}\,\Sigma^g \wedge \Sigma_g = 0\\ &\frac{\delta I}{\delta \Sigma^a} = F^a - \Psi^{-1}_{ae}\,\Sigma^e = 0 \;\longrightarrow\; F^a_{\mu\nu} = \Psi^{-1}_{ae}\,\Sigma^e_{\mu\nu} \end{aligned}\right\}. \tag{2}$$

The first equation of (2) states that A^g is the self-dual part of the spin connection compatible with the two forms Σ^a, where $\mathrm{D} = dx^\mu\,\mathrm{D}_\mu = dx^\mu\,(\partial_\mu + A_\mu)$ is the exterior covariant derivative with respect to A^a. The second equation implies that the two forms Σ^a can be constructed from tetrad one-forms $e^I = e^I_\mu dx^\mu$ in the form*

$$\Sigma^a = i\,e^0 \wedge e^a - \frac{1}{2}\,\epsilon_{afg}\,e^f \wedge e^g. \tag{3}$$

Equation (3) is a self-dual combination of tetrad wedge products, which enforces the equivalence of (1) to General Relativity. Note that equation (3) implies [14]

$$\frac{i}{2}\,\Sigma^a \wedge \Sigma^e = \delta^{ae}\sqrt{-g}\,d^4x\,, \tag{4}$$

*In the tetrad formulation of gravity, this corresponds to spacetimes of Lorentzian signature when e^0 is real, and Euclidean signature when e^0 is pure imaginary.

with the spacetime volume element as the proportionality factor. The third equation of motion in (2) states that the curvature of A^a is self-dual as a two form, which implies that the metric $g_{\mu\nu} = \eta_{IJ}\, e^I_\mu e^J_\nu$ derived from the tetrad one-forms e^I satisfies the vacuum Einstein equations.

If one were to eliminate the two forms Σ^a and the matrix ψ_{ae} from the action (1) while leaving the connection A^a_μ intact, then one would obtain the CDJ action [6], corresponding to the pure spin connection formulation of General Relativity. But we would like to obtain a formulation of General Relativity which preserves these fields to some extent, since they contain fundamental gravitational degrees of freedom and also provide a mechanism for implementing the initial value constraints.

The most direct way to preserve the ability to implement the constraints in a totally constrained system is to first perform a 3+1 decomposition of the action. The starting action (1) in component form is given by

$$I\left[\Sigma, A, \Psi\right] = \frac{1}{4}\int_{\mathrm{M}} d^4x \left(\Sigma^a_{\mu\nu} F^a_{\rho\sigma} - \frac{1}{2}\,\Psi^{-1}_{ae}\,\Sigma^a_{\mu\nu}\,\Sigma^e_{\rho\sigma}\right)\epsilon^{\mu\nu\rho\sigma}, \tag{5}$$

where $\epsilon^{0123} = 1$ and we have defined $\Psi^{-1}_{ae} = \delta_{ae}\,\varphi + \psi_{ae}$. For $\varphi = -\frac{\Lambda}{3}$, where Λ is the cosmological constant, then we have that

$$\Psi^{-1}_{ae} = -\frac{\Lambda}{3}\,\delta_{ae} + \psi_{ae}\,. \tag{6}$$

The matrix ψ_{ae}, presented in [5], is the self-dual part of the Weyl curvature tensor in SO(3,C) language. The eigenvalues of ψ_{ae} determine the algebraic classification of spacetime which is independent of coordinates and of tetrad frames [10, 11].* Ψ^{-1}_{ae} is the matrix inverse of Ψ_{ae} which we will refer to as the CDJ matrix, and is the result of appending to ψ_{ae} a trace part. In the CDJ formulation this field becomes eliminated in addition to the two forms Σ^a.

§2.1. The Ashtekar variables. Assuming a spacetime manifold of topology $\mathrm{M} = \Sigma \times \mathrm{R}$, where Σ refers to 3-space, let us perform a 3+1 decomposition of (5). Defining $\widetilde{\sigma}^i_a \equiv \frac{1}{2}\,\epsilon^{ijk}\,\Sigma^a_{jk}$ and $B^i_a \equiv \frac{1}{2}\,\epsilon^{ijk} F^a_{jk}$ for the spatial parts of the self-dual and curvature two forms, this is given by

$$I = \int dt \int_\Sigma d^3x\;\widetilde{\sigma}^i_a \dot{A}^a_i + A^a_0\,\mathrm{D}_i\,\widetilde{\sigma}^i_a + \Sigma^a_{0i}\left(B^i_a - \Psi^{-1}_{ae}\,\widetilde{\sigma}^i_e\right), \tag{7}$$

where we have integrated by parts, using $F^a_{0i} = \dot{A}^a_i - \mathrm{D}_i A^a_0$ from the tem-

*This includes principal null directions and properties of gravitational radiation.

poral component of the curvature. The operator D_i is the spatial part of the SO(3,C) covariant derivative, which in (1) acts as a covariant divergence. The following action ensues on any SO(3,C)-valued vector v_a, given by $\mathrm{D}_i v_a = \partial_i v_a + f_{abc} A^b_i v_c$. We will use (2) and (3) to redefine the two form components in (7).

Define e^a_i as the spatial part of the tetrads e^I_μ and make the identification

$$e^a_i = \frac{1}{2}\,\epsilon_{ijk}\,\epsilon^{abc}\,\widetilde{\sigma}^j_b\,\widetilde{\sigma}^k_c\,(\det\|\widetilde{\sigma}\|)^{-1/2} = \sqrt{\det\|\widetilde{\sigma}\|}\,\left(\widetilde{\sigma}^{-1}\right)^a_i\,. \tag{8}$$

For a special case $e^0_i = 0$, known as the time gauge, then the temporal components of the two forms (3) are given by (see e.g. [13, 15])

$$\Sigma^a_{0i} = \frac{i}{2}\,\underline{N}\,\epsilon_{ijk}\,\epsilon^{abc}\,\widetilde{\sigma}^j_b\,\widetilde{\sigma}^k_c + \epsilon_{ijk}\,N^j\,\widetilde{\sigma}^k_a\,, \tag{9}$$

where $\underline{N} = N\,(\det\|\widetilde{\sigma}\|)^{-1/2}$ with N and N^i being a set of four nondynamical fields. In the steps leading to the CDJ action of [6], the fields $N^\mu = (N, N^i)$ become eliminated along with the process of eliminating the 2-forms $\Sigma^a_{\mu\nu}$.

Substituting (9) into (7), we obtain the action

$$I = \int dt \int_\Sigma d^3x\;\widetilde{\sigma}^i_a\,\dot{A}^a_i + A^a_0\,G_a - N^i H_i - iNH\,. \tag{10}$$

The fields (A^a_0, N, N^i) are auxiliary fields whose variations yield respectively the following constraints

$$\left.\begin{aligned} G_a &= \mathrm{D}_i\,\widetilde{\sigma}^i_a \\ H_i &= \epsilon_{ijk}\,\widetilde{\sigma}^j_a\,B^k_a + \epsilon_{ijk}\,\widetilde{\sigma}^j_a\,\widetilde{\sigma}^k_e\,\Psi^{-1}_{ae} \\ H &= (\det\|\widetilde{\sigma}\|)^{-1/2}\,\times \\ &\times\left[\frac{1}{2}\,\epsilon_{ijk}\,\epsilon^{abc}\,\widetilde{\sigma}^i_a\,\widetilde{\sigma}^j_b\,B^k_c - \frac{1}{6}\left(\mathrm{tr}\,\Psi^{-1}\right)\epsilon_{ijk}\,\epsilon_{abc}\,\widetilde{\sigma}^i_a\,\widetilde{\sigma}^j_b\,\widetilde{\sigma}^k_c\right] \end{aligned}\right\}. \tag{11}$$

Rather than attempt to perform a canonical analysis, we will proceed from (10) as follows. Think of $I = I_{\widetilde{\sigma},\Psi}[A]$ as an infinite dimensional functional manifold of theories parametrized by the fields $\widetilde{\sigma}^i_a$ and Ψ_{ae}, and then restrict attention to a submanifold corresponding to the theory of General Relativity.

Following suit, say that we impose the following conditions on Ψ^{-1}_{ae}

$$\epsilon^{bae}\,\Psi^{-1}_{ae} = 0\,, \qquad \mathrm{tr}\,\Psi^{-1} = -\Lambda \tag{12}$$

with no restrictions on $\widetilde{\sigma}^i_a$, where Λ is the cosmological constant. Then Ψ^{-1}_{ae} becomes eliminated and equation (10) reduces to the action for General Relativity in the Ashtekar variables (see e.g. [1–3])

$$I_{\text{Ash}} = \frac{i}{G}\int dt \int_\Sigma d^3x\, \widetilde{\sigma}^i_a \dot{A}^a_i + A^a_0 \mathrm{D}_i \widetilde{\sigma}^i_a - \\ - \epsilon_{ijk} N^i \widetilde{\sigma}^j_a B^k_a + \frac{i}{2}\, \underline{N} \epsilon_{ijk} \epsilon_{abc} \widetilde{\sigma}^i_a \widetilde{\sigma}^j_b \left(B^k_c + \frac{\Lambda}{3} \widetilde{\sigma}^k_c \right), \tag{13}$$

where $\underline{N} = N\left(\det\|\widetilde{\sigma}\|\right)^{-1/2}$ is the densitized lapse function. The action (13) is written on the phase space $\Omega_{\text{Ash}} = (\widetilde{\sigma}^i_a, A^a_i)$ and the variable Ψ^{-1}_{ae} has been eliminated. The auxiliary fields A^a_0, N and N^i respectively are the SO(3,C) rotation angle, the lapse function and the shift vector. These auxiliary fields play the role of Lagrange multipliers smearing their associated initial value constraints G_a, H, and H_i, respectively the Gauss' law, Hamiltonian and vector (sometimes known as diffeomorphism) constraints. Note that $\widetilde{\sigma}^i_a$ in the original Plebanski action was part of an auxiliary field $\Sigma^a_{\mu\nu}$, but now in (13) it has become promoted to the status of a momentum space dynamical variable. At the level of (13), one could further eliminate the 2-forms Σ^a to obtain the CDJ pure spin connection action appearing in [6]. However, (13) is already in a form suitable for quantization and for implementation of the initial value constraints via the temporal parts of these 2-forms.

§3. The instanton representation. Having shown that Plebanski's action (1) contains (13), an action known to describe General Relativity, as a direct consequence of (12), we will now show that (1) also contains an alternate formulation of General Relativity based on the field Ψ_{ae}, which can also be derived directly from (5).

Instead of equation (12), let us impose the following conditions in the constraints (11)

$$\epsilon_{ijk} \epsilon_{abc} \widetilde{\sigma}^i_a \widetilde{\sigma}^j_b B^k_c = -\frac{\Lambda}{3} \epsilon_{ijk} \epsilon_{abc} \widetilde{\sigma}^i_a \widetilde{\sigma}^j_b \widetilde{\sigma}^k_c\,, \qquad \epsilon_{ijk} \widetilde{\sigma}^j_a B^k_a = 0 \tag{14}$$

with no restriction on Ψ_{ae}. Substitution of (14) into (11) yields

$$\left.\begin{aligned} & H_i = \epsilon_{ijk} \widetilde{\sigma}^j_a \widetilde{\sigma}^k_e \Psi^{-1}_{ae} \\ & H = \left(\det\|\widetilde{\sigma}\|\right)^{-1/2} \left[-\frac{\Lambda}{6} \epsilon_{ijk} \epsilon_{abc} \widetilde{\sigma}^i_a \widetilde{\sigma}^j_b \widetilde{\sigma}^k_c - \right. \\ & \left. - \frac{1}{6} \left(\operatorname{tr}\Psi^{-1}\right) \epsilon_{ijk} \epsilon_{abc} \widetilde{\sigma}^i_a \widetilde{\sigma}^j_b \widetilde{\sigma}^k_c \right] = -\sqrt{\det\|\widetilde{\sigma}\|}\left(\Lambda + \operatorname{tr}\Psi^{-1}\right) \end{aligned}\right\}. \tag{15}$$

Hence substituting (15) into (10), we obtain an action given by

$$I = \int dt \int_\Sigma d^3x \, \widetilde{\sigma}^i_a \dot{A}^a_i + A^a_0 \mathrm{D}_i \widetilde{\sigma}^i_a + \\ + \epsilon_{ijk} N^i \widetilde{\sigma}^j_a \widetilde{\sigma}^k_e \Psi^{-1}_{ae} - iN\sqrt{\det\|\widetilde{\sigma}\|} \left(\Lambda + \mathrm{tr}\,\Psi^{-1}\right). \qquad (16)$$

But (16) still contains $\widetilde{\sigma}^i_a$, therefore we will completely eliminate $\widetilde{\sigma}^i_a$ by substituting the spatial restriction of the third equation of motion of (2), given by

$$\widetilde{\sigma}^i_a = \Psi_{ae} B^i_e \,, \qquad (17)$$

into (16). This substitution, which also appears in [6] in the form of the so-called CDJ ansatz, yields the action*

$$I_{\mathrm{Inst}} = \int dt \int_\Sigma d^3x \, \Psi_{ae} B^i_e \dot{A}^a_i + A^a_0 B^i_e \mathrm{D}_i \Psi_{ae} + \\ + \epsilon_{ijk} N^i B^j_a B^k_e \Psi_{ae} - iN\sqrt{\det\|B\|}\sqrt{\det\|\Psi\|} \left(\Lambda + \mathrm{tr}\,\Psi^{-1}\right), \quad (18)$$

which depends on the CDJ matrix Ψ_{ae} and the Ashtekar connection A^a_i, with no appearance of $\widetilde{\sigma}^i_a$. In the original Plebanski theory Ψ_{ae} was an auxiliary field which could be eliminated. But now Ψ_{ae} has become promoted to the status of a full dynamical variable, analogously to the case for $\widetilde{\sigma}^i_a$ in I_{Ash}.

There are a few items of note regarding (18). Note that it contains the same auxiliary fields (A^a_0, N, N^i) as in the Ashtekar theory (13). Since we have imposed the constraints $H_\mu = (H, H_i)$ on the Ashtekar phase space within the starting Plebanski theory in order to obtain I_{Inst}, then this suggests that the initial value constraints (G_a, H, H_i) should play an analogous role in (18) as their counterparts in (13). This relation holds only when Ψ_{ae} is nondegenerate, which limits one to spacetimes of Petrov Types I, D and O where Ψ_{ae} has three linearly independent eigenvectors.† Lastly, note that by further elimination of Ψ_{ae} and N^i from (18) one can obtain the CDJ action in [6]. However, we would like to preserve Ψ_{ae} since it contains gravitational degrees of freedom relevant to the instanton representation, and the shift vector N^i as we will see also assumes an important role.

*Equation (17) is valid when B^i_a and Ψ_{ae} are nondegenerate as 3×3 matrices. Hence all results of this paper will be confined to configurations where this is the case.

†We refer to (18) as the instanton representation of Plebanski gravity because it follows directly from Plebanski's action (1). We will in this sense use (18) as the starting point for the reformulation of gravity thus presented. The association of (18) with gravitational instantons will be made more precise later in this paper.

§4. Equations of motion of the instanton representation. We will now show that Einstein equations follow from the instanton representation action I_{Inst} in the same sense that they follow from the original Plebanski action (1). More precisely, we will demonstrate consistency of the equations of motion of (18) with equations (2) and (3). After integrating by parts and discarding boundary terms, the starting action (18) is given by

$$I_{\text{Inst}} = \int dt \int_{\Sigma} d^3x \; \Psi_{ae} B^k_e \left(F^a_{0k} + \epsilon_{kjm} B^j_a N^m\right) - \\ - iN\sqrt{\det\|B\|}\sqrt{\det\|\Psi\|}\left(\Lambda + \operatorname{tr}\Psi^{-1}\right). \tag{19}$$

The equation of motion for the shift vector N^i is given by

$$\frac{\delta I_{\text{Inst}}}{\delta N^i} = \epsilon_{ijk} B^j_a B^k_e \Psi_{ae} = 0\,, \tag{20}$$

which implies on the solution to the equations of motion that $\Psi_{ae} = \Psi_{(ae)}$ is symmetric.

The equation of motion for the lapse function N is given by

$$\frac{\delta I_{\text{Inst}}}{\delta N} = \sqrt{\det\|B\|}\sqrt{\det\|\Psi\|}\left(\Lambda + \operatorname{tr}\Psi^{-1}\right) = 0\,. \tag{21}$$

Nondegeneracy of Ψ_{ae} and of the magnetic field B^i_e implies that on-shell, the following relation must be satisfied

$$\Lambda + \operatorname{tr}\Psi^{-1} = 0\,, \tag{22}$$

which implies that λ_3 can be written explicitly in terms of λ_1 and λ_2, regarded as physical degrees of freedom. The equation of motion for Ψ_{ae} is

$$\frac{\delta I_{\text{Inst}}}{\delta \Psi_{ae}} = B^k_e F^a_{0k} + \epsilon_{kjm} B^k_e B^j_a N^m + \\ + iN\sqrt{\det\|B\|}\sqrt{\det\|\Psi\|}\left(\Psi^{-1}\Psi^{-1}\right)^{ea} = 0\,, \tag{23}$$

where we have used (22). The symmetric and the antisymmetric parts of (23) must separately vanish. The antisymmetric part is given by

$$B^k_{[e} F^{a]}_{0k} + \epsilon_{mkj} N^m B^k_e B^j_a = 0\,, \tag{24}$$

which can be used to solve for the shift vector N^i. Using the relation $\epsilon_{ijk} B^j_a B^k_e = \epsilon_{aed}\,(B^{-1})^d_i\,(\det\|B\|)$ for nondegenerate 3×3 matrices, we have

$$N^i = -\frac{1}{2}\,\epsilon^{ijk} F^g_{0j}\left(B^{-1}\right)^{\,g}_{k}. \tag{25}$$

The symmetric part of (23) is given by

$$B^k_{(e}F^{a)}_{0k} + iN\sqrt{\det\|B\|}\sqrt{\det\|\Psi\|}\,\big(\Psi^{-1}\Psi^{-1}\big)^{(ea)} = 0\,, \tag{26}$$

where we have used that Ψ_{ae} on-shell is symmetric from (20).

§4.1. Verification of the Einstein equations. To make a direct connection from the instanton representation to Einstein's General Relativity, we will show that the equations of motion for I_{Inst} imply the Einstein equations. Let us use the relation

$$\sqrt{-g} = N\sqrt{h} = N\sqrt{\det\|\widetilde{\sigma}\|} = \sqrt{\det\|B\|}\sqrt{\det\|\Psi\|}\,, \tag{27}$$

which writes the determinant of the spacetime metric $g_{\mu\nu}$ in terms of dynamical variables (A,Ψ) using the 3+1 decomposition, and uses the determinant of (17). Defining $\epsilon^{0ijk}\equiv\epsilon^{ijk}$ and using the symmetries of the four-dimensional epsilon tensor $\epsilon^{\mu\nu\rho\sigma}$, then the following identities hold

$$B^k_{(e}F^{a)}_{0k} = \frac{1}{2}\,\epsilon^{klm}F^{(e}_{lm}F^{a)}_{0k} = \frac{1}{8}\,\epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^e_{\rho\sigma}\,. \tag{28}$$

Using (28) and (27), then equation (26) can be re-written as

$$\frac{1}{8}\,F^b_{\mu\nu}F^f_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} + i\sqrt{-g}\,\big(\Psi^{-1}\Psi^{-1}\big)^{(bf)} = 0\,. \tag{29}$$

Left and right multiplying (29) by Ψ, which is symmetric after implementation of (20), we obtain

$$\frac{1}{4}\,\big(\Psi^{bb'}F^{b'}_{\mu\nu}\big)\big(\Psi^{ff'}F^{f'}_{\rho\sigma}\big)\,\epsilon^{\mu\nu\rho\sigma} = -\,2i\sqrt{-g}\,\delta^{bf}. \tag{30}$$

Note that this step and the steps that follow require that Ψ_{ae} be nondegenerate as a 3×3 matrix. Let us make the definition

$$\Sigma^a_{\mu\nu} = \Psi_{ae}F^e_{\mu\nu} = \Sigma^a_{\mu\nu}\big[\Psi,A\big], \tag{31}$$

which retains Ψ_{ae} and A^a_μ as fundamental, with the two forms $\Sigma^a_{\mu\nu}$ being derived quantities. Upon using the third line of (2) as a re-definition of variables, which amounts to using the curvature and the CDJ matrix to construct a two form, (30) reduces to

$$\frac{1}{4}\,\Sigma^b_{\mu\nu}\Sigma^f_{\rho\sigma}\,dx^\mu\wedge dx^\nu\wedge dx^\rho\wedge dx^\sigma = \Sigma^b\wedge\Sigma^f = -\,2i\sqrt{-g}\,\delta^{bf}d^4x\,. \tag{32}$$

One recognizes (32) as the condition that the two forms thus constructed, which are now derived quantities, be derivable from tetrads, which is the analogue of (4). Indeed, one can conclude as a consequence of (32) that there exist one forms $e^I = e^I_\mu dx^\mu$ where $I = 0, 1, \dots 3$, such that

$$\Psi_{ae} F^e = ie^0 \wedge e^a - \frac{1}{2}\,\epsilon_{afg}\, e^f \wedge e^g \equiv P^a_{fg}\, e^f \wedge e^g. \tag{33}$$

We have defined P^a_{fg} as a projection operator onto the self-dual combination of one-form wedge products, self-dual in the SO(3,C) sense. To complete the demonstration that the instanton representation yields the Einstein equations, it remains to show that the connection A^a is compatible with the two forms Σ^a as constructed in (31).

Using the fact that Ψ_{ae} is symmetric on solutions to (20), the starting action (19) can be written as*

$$I_{\text{Inst}} = \int_{\text{M}} d^4x\, \frac{1}{8}\, \Psi_{ae} F^a_{\mu\nu} F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} - i\sqrt{-g}\,\big(\Lambda + \text{tr}\,\Psi^{-1}\big)\,. \tag{34}$$

The equation of motion for the connection A^a_μ from (34) is given by

$$\frac{\delta I_{\text{Inst}}}{\delta A^a_\mu} \sim \epsilon^{\mu\sigma\nu\rho} \text{D}_\sigma (\Psi_{ae} F^e_{\nu\rho}) - \\ - \frac{i}{2}\, \delta^\mu_i \text{D}^{ij}_{da} \Big[N \big(B^{-1}\big)^d_j \sqrt{\det\|B\|}\sqrt{\det\|\Psi\|}\,\big(\Lambda + \text{tr}\,\Psi^{-1}\big)\Big]\,, \tag{35}$$

where we have used that Ψ_{ae} is symmetric and we have defined

$$\left.\begin{aligned} \overline{\text{D}}^{ji}_{ea}(x,y) &\equiv \frac{\delta B^j_e(y)}{\delta A^a_i(x)} = \epsilon^{jki}\big(-\delta_{ae}\,\partial_k + f_{eda} A^d_k\big)\,\delta^{(3)}(x,y) \\ \overline{\text{D}}^{0i}_{ea} &\equiv 0 \end{aligned}\right\}. \tag{36}$$

The term in square brackets in (35) vanishes on-shell, since it is proportional to the equation of motion (21) and its spatial derivatives, which leaves us with

$$\epsilon^{\mu\sigma\nu\rho}\, \text{D}_\sigma \big(\Psi_{ae} F^e_{\nu\rho}\big) = 0\,. \tag{37}$$

Equation (37) states that when (20) and (22) are satisfied, then the two forms $\Sigma^a_{\mu\nu}$ constructed from Ψ_{ae} and $F^e_{\mu\nu}$ as in (31) are compatible with the connection A^a_μ. This is the direct analogue of the first equation from (2).

Using (19) as the starting point, which uses Ψ_{ae} and A^a_μ as the dynamical variables, we have obtained the Einstein equations in the same

*The same action was written down in [6], which arises from elimination of the self-dual 2-forms directly from Plebanski's action. In the approach of the present paper, we have eliminated only the spatial part of the 2-forms, and have used the antisymmetric part of Ψ_{ae} to solve for the shift vector N^i.

sense that the starting Plebanski theory (1) implies the Einstein equations. The first equation of (2) has been reproduced via (37), which holds provided that (22) and (20) are satisfied. The second equation of (2) has been reproduced via (32), which follows from (29) when (20) is satisfied. The third equation of (2) may be regarded as a defining relation for the instanton representation. Since the Einstein equations have arisen from the instanton representation, then it follows that I_{Inst} is another representation for General Relativity for nondegenerate Ψ_{ae} and B^i_e.

On the solution to (20) and (22) and using (33), the action for the instanton representation can be written in the language of two forms as

$$I_{\text{Inst}} = \frac{1}{2}\int_{\text{M}} \Psi_{bf} F^b \wedge F^f = \frac{1}{2}\int_{\text{M}} P^a_{fg}\, e^f \wedge e^g \wedge F^a, \tag{38}$$

which upon the identification of one forms e^I with tetrads, is nothing other than the self-dual Palatini action [17].

Note that the Palatini action implies the Einstein equations with respect to the metric defined by

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = \eta_{IJ}\, e^I \otimes e^J, \tag{39}$$

where η_{IJ} is the Minkowski metric, which provides additional confirmation that the instanton representation I_{Inst} describes Einstein's General Relativity when Ψ_{ae} is nondegenerate.

§4.2. Discussion: constructing a solution. We have shown how the Einstein equations follow from the instanton representation (18), which uses Ψ_{ae} and A^a_μ as the dynamical variables. Equation (30) implies the existence of a tetrad, which imposes equivalence of I_{Inst} with General Relativity, but it does not explain how to construct the tetrad. Since the spacetime metric $g_{\mu\nu}$ is the fundamental variable in Einstein's theory, we will bypass the tetrad and construct $g_{\mu\nu}$ directly as follows.

Perform a 3+1 decomposition of spacetime $\text{M} = \Sigma \times \text{R}$, where Σ is a three-dimensional spatial hypersurface. The line element is given by

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}\, \omega^i \otimes \omega^j, \tag{40}$$

where h_{ij} is the induced 3-metric on Σ, and we have defined the one form

$$\omega^i = dx^i - N^i dt. \tag{41}$$

The shift vector is given by (25), rewritten here for completeness

$$N^i = -\frac{1}{2}\, \epsilon^{ijk} F^g_{0j} \big(B^{-1}\big)^{\,g}_{k}, \tag{42}$$

and the lapse function N can apparently be chosen freely.

To complete the construction of $g_{\mu\nu}$ using I_{Inst} as the starting point we must write the 3-metric h_{ij} using Ψ_{ae} and A^a_μ. The desired expression is given by

$$h_{ij} = (\det\|\Psi\|)\big(\Psi^{-1}\Psi^{-1}\big)^{ae}\big(B^{-1}\big)^a_i\big(B^{-1}\big)^e_j(\det\|B\|) = h_{ij}\big[\Psi, A\big], \quad (43)$$

where the following conditions must be satisfied

$$B^i_e \,\mathrm{D}_i \Psi_{ae} = 0\,, \quad \epsilon_{dae}\Psi_{ae} = 0\,, \quad \Lambda + \mathrm{tr}\,\Psi^{-1} = 0\,. \quad (44)$$

Equations (44) will be referred to as the Gauss' law, diffeomorphism and Hamiltonian constraints, which follow from variation of Lagrange multipliers A^a_0, N^i and N in the action (18). Note that equations (44) involve only Ψ_{ae} and the spatial spart of the connection A^a_μ, objects which determine a spatial metric in (43).

The spacetime metric $g_{\mu\nu}$ solving the Einstein equations is given by

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^i N_i & -N_j \\ -N_i & h_{ij} \end{pmatrix},$$

where $N_i = h_{ik}N^k$. There are a few things to note regarding this:

1) From (42), the shift vector N^i depends only on A^a_μ, which contains gauge degrees of freedom in the temporal component A^a_0;
2) Secondly, the lapse function N is freely specifiable;
3) Third, each A^a_i and Ψ_{ae} satisfying the initial value constraints (44) determines a 3-metric h_{ij}, which when combined with a choice of A^a_0 and lapse function N should provide a solution $g_{\mu\nu}$ for spacetimes of Petrov Type I, D and O.

Note, when one uses the CDJ ansatz $\widetilde{\sigma}^i_a = \Psi_{ae}B^i_e$ that (43) implies $hh^{ij} = \widetilde{\sigma}^i_a\widetilde{\sigma}^j_a$, which is the relation of the Ashtekar densitized triad to the contravariant 3-metric h^{ij} [1]. Upon implementation of (44) on the phase space Ω_{Inst}, then one is left with the two degrees of freedom per point of General Relativity, and h_{ij} becomes expressed explicitly in terms of these degrees of freedom.

§5. Analysis of the equations of motion.

We will now provide a rudimentary analysis of the physical content of the equations of motion of I_{Inst} beyond the Einstein equations. The first equation, re-written here for completeness, is (23)

$$B^i_f F^b_{0i} + i\sqrt{-g}\,\big(\Psi^{-1}\Psi^{-1}\big)^{fb} + \epsilon_{ijk}B^i_f B^j_b N^k = 0\,. \quad (45)$$

Also, when (20) and (22) are satisfied, then (35) implies (37), also written here

$$\epsilon^{\mu\sigma\nu\rho}\,\mathrm{D}_\sigma\big(\Psi_{ae}F^e_{\nu\rho}\big)=0\,. \tag{46}$$

We have shown that when Ψ_{bf} is symmetric after determination of N^i as in (25), that the symmetric part of (45) in conjunction with (46) imply the Einstein equations. We will now show under this condition that (45) and (46) form a self-consistent system. Act on (46) with D_μ and use the definition of curvature as the commutator of covariant derivatives, yielding

$$\epsilon^{\mu\nu\rho\sigma}\,\mathrm{D}_\mu\mathrm{D}_\nu\left(\Psi_{ae}F^e_{\rho\sigma}\right)=f_{abc}\,\Psi_{ce}\left(F^b_{\mu\nu}F^e_{\rho\sigma}\,\epsilon^{\mu\nu\rho\sigma}\right)=0\,. \tag{47}$$

Then substituting the symmetric part of (45) into (47), up to an insignificant numerical factor we get

$$f_{abc}\,\Psi_{ce}\left[\,i\sqrt{-g}\,\left(\Psi^{-1}\Psi^{-1}\right)^{(eb)}\,\right]=i\sqrt{-g}\,f_{abc}\left(\Psi^{-1}\right)^{cb}=0\,, \tag{48}$$

which is simply the statement that Ψ_{ce} is symmetric in c and e which is consistent with (20) for $\det\|B\|\neq 0$. This can also be seen at the level of 2-forms by elimination of the curvature from (47) to obtain

$$f_{abc}\,F^b_{\mu\nu}\,\Sigma^c_{\rho\sigma}\,\epsilon^{\mu\nu\rho\sigma}\longrightarrow f_{abc}\,(\Psi^{-1})^{bf}\,\Sigma^f_{\mu\nu}\,\Sigma^c_{\rho\sigma}\,\epsilon^{\mu\nu\rho\sigma}\sim 0 \tag{49}$$

due to (32), on account of antisymmetry of the structure constants.

We will now multiply (45) by $(B^{-1})^f_k$, in conjunction with using the identity $(B^{-1})^d_j B^j_b=\delta^d_b$ since B^i_f is nondegenerate. Then equation (45) can be written as

$$\begin{aligned}F^b_{0k}+iN\left(\det\|B\|\right)^{-1/2}&\left(\det\|\Psi\|\right)^{-1/2}\left(\det\|B\|\det\|\Psi\|\right)\times\\ &\times\left[(\Psi^{-1}\Psi^{-1})^{df}(B^{-1})^d_j(B^{-1})^f_k\right]B^j_b+\epsilon_{kjm}B^j_bN^m=0\,.\end{aligned} \tag{50}$$

We can now use (43) in the second term of (50), which defines the spatial 3-metric in terms of Ψ_{ae} and the spatial connection A^a_i solving the constraints (44). Using this in conjunction with the relation $N(\det\|B\|)^{-1/2}(\det\|\Psi\|)^{-1/2}=Nh^{-1/2}=\underline{N}$, then equation (50) becomes

$$F^b_{0i}+i\underline{N}\,h_{ij}\,B^j_b+\epsilon_{ijk}\,B^j_b\,N^k=0\,. \tag{51}$$

We will show in the next subparagraph that (51) is simply the statement that the curvature $F^a_{\mu\nu}$ is Hodge self-dual with respect to a metric $g_{\mu\nu}$ whose spatial part is h_{ij}, whose lapse function is N and whose shift vector is N^i.

It may appear via (30) that only the symmetric part of (45) is needed in order for I_{Inst} to imply the Einstein equations for Petrov Types I, D and O. But we have utilized the equation of motion (45) to arrive at (51), which includes information derived using the antisymmetric part of Ψ_{ae}. The reconciliation is in the observation that part of the process of solving the Einstein equations involves computing the shift vector via (25), which simultaneously eliminates the antisymmetric part of (45). Since (51) then is consistent with the Einstein equations, then the implication is that each such solution is included within the class of configurations under which the curvature $F^a_{\mu\nu}$ is Hodge self-dual with respect to the corresponding metric $g_{\mu\nu}$. The spatial part h_{ij} of this metric is defined on the configurations (Ψ_{ae}, A^a_i) satisfying (44).

§5.1. Dynamical Hodge self-duality operator. We will now prove that equation (51) is indeed the statement that the curvature $F^a_{\mu\nu}$ is Hodge self-dual with respect to $g_{\mu\nu} = g_{\mu\nu}[\Psi, A]$. To show this, we will derive the Hodge self-duality condition for Yang-Mills theory in curved spacetime, using the 3+1 decomposition of the associated metric.

The following relations will be useful

$$g^{00} = -\frac{1}{N^2}\,, \qquad g^{0i} = -\frac{N^i}{N^2}\,, \qquad g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}\,, \tag{52}$$

where N is real for Lorentzian signature spacetimes and pure imaginary for Euclidean signature.

The Hodge self-duality condition for the curvature $F^a_{\mu\nu}$ can be written in the following form

$$\sqrt{-g}\, g^{\mu\rho} g^{\nu\sigma} F^a_{\rho\sigma} = \frac{\beta}{2}\, \epsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma}\,, \tag{53}$$

where β is a numerical constant which we will determine. Expanding (53) and using $F^a_{00} = 0$, we have

$$N\sqrt{h}\left[\left(g^{\mu 0} g^{\nu j} - g^{\nu 0} g^{\mu j}\right) F^a_{0j} + g^{\mu i} g^{\nu j} \epsilon_{ijk} B^k_a\right] = \\ = \frac{\beta}{2}\left(2\,\epsilon^{\mu\nu 0 i} F^a_{0i} + \epsilon^{\mu\nu ij} \epsilon_{ijm} B^m_a\right). \tag{54}$$

We will now examine the individual components of (54). The $\mu = 0$, $\nu = 0$ component yields $0 = 0$, which is trivially satisfied. Moving on to the $\mu = 0$, $\nu = k$ component, we have

$$N\sqrt{h}\left[\left(g^{00} g^{kj} - g^{k0} g^{0j}\right) F^a_{0j} + g^{0i} g^{kj} \epsilon_{ijm} B^m_a\right] = \beta B^k_a\,. \tag{55}$$

Making use of (52) as well as the antisymmetry of the epsilon symbol, after some algebra* we obtain

$$F^a_{0j} + \epsilon_{jmk} B^m_a N^k + \beta \underline{N} h_{jk} B^k_a = 0\,, \tag{56}$$

where we have defined $\underline{N} = N h^{-1/2}$. Note that (56) is the same as (51) for $\beta = i$, which establishes Hodge self-duality with respect to the spatio-temporal components.

We must next verify Hodge self-duality with respect to the purely spatial components of the curvature. For the $\mu = m$, $\nu = n$ component of (53), we have

$$N\sqrt{h}\left[\left(g^{m0}g^{nj} - g^{n0}g^{mj}\right)F^a_{0j} + g^{mi}g^{nj}\epsilon_{ijk}B^k_a\right] = \beta\,\epsilon^{mn0i}F^a_{0i}\,. \tag{57}$$

Substitution of (52) into (57) after some algebra yields†

$$\frac{\sqrt{h}}{N}\left(N^n h^{mj} - N^m h^{nj}\right)\left(F^a_{0j} + \epsilon_{jkl}B^k_a N^l\right) = \\ = \epsilon^{mnl}\left(\beta F^a_{0l} - \underline{N} h_{lk} B^k_a\right). \tag{58}$$

Using $h^{ij}h_{jk} = \delta^i_k$ and simplifying, then (58) reduces to

$$F^a_{0k} + \epsilon_{kmn} B^m_a N^n = \frac{1}{\beta}\,\underline{N} h_{kl} B^l_a\,. \tag{59}$$

Consistency of (59) with (56) implies that $\frac{1}{\beta} = -\beta$, or $\beta = \pm i$. Comparison of (56) and (59) with (51) shows that the Hodge self-duality condition arises dynamically from the equations of motion (18) of I_{Inst}. Moreover, the curvature $F^a_{\mu\nu}$ is Hodge self-dual with respect to this operator, which can be written as‡

$$H^{\mu\nu\rho\sigma}_{\pm} = \frac{1}{2}\left[\sqrt{-g}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma}\right) \pm i\,\epsilon^{\mu\nu\rho\sigma}\right], \tag{60}$$

where $g_{\mu\nu} = g_{\mu\nu}[\Psi, A]$ is defined in terms of instanton representation variables.

The results can then be summarized as follows. The instanton representation I_{Inst} on-shell implies that the SO(3,C) gauge curvature $F^a_{\mu\nu}$

---

*See Appendix A leading to equation (130).

†See Appendix A leading to equation (138).

‡It appears that $\beta = \pm i$ follows from our choice of a Lorentzian signature metric corresponding to real N, and that one can make a Wick rotation $N \to iN$, and analogously require $\beta = \pm 1$ for Euclidean signature. However, we will show in this paper that the reality conditions play a role in the signature of spacetime, more so than does the choice of lapse function N.

is Hodge self-dual with respect to a metric $g_{\mu\nu}$. But I_{Inst} also implies on-shell that $g_{\mu\nu}$ solves the Einstein equations, which in turn identifies $F^a_{\mu\nu}$ with the Riemann curvature $\mathfrak{Riem} \equiv R_{\mu\nu\rho\sigma}$. Hence $\mathfrak{Riem}$ is also Hodge self-dual on any solution, which implies that the solutions of I_{Inst} correspond to gravitational instantons.*

§6. Relation to the CDJ pure spin connection formulation. There is an action for General Relativity derived by Capovilla, Dell and Jacobson (CDJ), which can be written almost entirely in terms of the spin connection [6]. The authors used Plebanski's action (1) as the starting point, from which they proceed to eliminate the 2-forms $\Sigma^a_{\mu\nu}$ and the matrix ψ_{ae}, leading for $\Lambda = 0$ to the action

$$I_{\text{CDJ}} = \int_{\text{M}} d^4x \ \text{tr}\left[M\left(M - \frac{1}{2}\,\text{tr}\,M\right)\right], \tag{61}$$

where we have defined

$$M^{bf} = -\frac{i}{8\sqrt{-g}}\,F^b_{\mu\nu}F^f_{\rho\sigma}\,\epsilon^{\mu\nu\rho\sigma}. \tag{62}$$

Note that equation (62) for [6] is the same as (29), which is the symmetric part of (51). The action (34) serves in [6] as an intermediate step in obtaining the action (61) from (1).† But in our context, equation (34) follows from (18) after elimination of $\Psi_{[ae]}$ and N^i through their equations of motion.

Given that the CDJ action essentially follows from (18) after elimination of Ψ_{ae}, then this implies that Ψ_{ae} should satisfy equation (2.20b) of [6] on any solution for $\Lambda = 0$. We will show this by following the same steps in [6]. To obtain Ψ_{ae} in terms of A^a_μ, one would need to take the square root of M^{bf} in (62). This introduces various complications, which are circumvented in [6] by using the characteristic equation for (a symmetric) Ψ_{ae}

$$\Psi^{-3} - \left(\text{tr}\,\Psi^{-1}\right)\Psi^{-2} + \frac{1}{2}\left[\left(\text{tr}\,\Psi^{-1}\right)^2 - \text{tr}\,\Psi^{-2}\right]\Psi^{-1} - \det\|\Psi\|^{-1} = 0\,. \tag{63}$$

---

*The gauge curvature $F^a_{\mu\nu}$ takes its values in the SO(3,C) Lie algebra corresponding to the self-dual half of the Lorentz group SO(3,1). The equivalence of internal self-duality with Hodge self-duality makes sense when one has a tetrad e^I_μ, which intertwines between internal and spacetime indices. But since tetrads are now derived quantities in I_{Inst}, this feature appears to be more fundamentally related to the Yang-Mills aspects of the theory. We will show in a few paragraphs that this is indeed the case.

†Note that Ψ in the present paper, after the elimination of the shift vector N^i is actually defined as Ψ^{-1} in [6].

One must then use $\Psi^{-1}\Psi^{-1}=M$ from (29) as well as $\mathrm{tr}\,\Psi^{-1}=-\Lambda$ from (22), which when substituted into (63) yields the equation

$$\left[M+\frac{1}{2}\left(-\mathrm{tr}\,M+\Lambda^2\right)\right]\Psi^{-1}=-\Lambda M+I\sqrt{\det\|M\|}\,, \tag{64}$$

where I is the unit 3×3 matrix. Then assuming that the left hand side of (64) is invertible, one can solve for $\Psi_{(ae)}$ as

$$\Psi_{(ae)}=\left(-\Lambda_{af}+\delta_{af}\sqrt{\det\|M\|}\right)^{-1}\left[M_{fe}+\frac{1}{2}\,\delta_{fe}\left(\Lambda^2-\mathrm{tr}\,M\right)\right]. \tag{65}$$

Then upon substitution of (65) into (34) one obtains the CDJ action (62) for $\Lambda=0$. For $\Lambda\neq0$ one can expand (65) in powers of Λ using a geometric series, yielding

$$\Psi_{ae}=-\frac{1}{\Lambda}\left\{\left[\delta_{ae}-\frac{\Lambda\left(\Lambda^2-\mathrm{tr}\,M\right)}{2\sqrt{\det\|M\|}}+1\right]\left[\delta_{ae}-\frac{\Lambda M_{ae}}{\sqrt{\det\|M\|}}\right]^{-1}\right\}. \tag{66}$$

Then one obtains the analogue of equation (3.9) of [6], which we will not display here.

Let us now comment on the differences between (18) and (34), namely equation (2.8) in [6]. Equation (34) can be obtained by elimination of the 2-forms $\Sigma^a_{\mu\nu}$ directly from (1). Then the CDJ action (61) follows by further elimination of the field Ψ_{ae}. But (18) is the result of eliminating only Σ^a_{ij}, the spatial part of the 2-forms, and preserving the temporal components Σ^a_{0i} as well as Ψ_{ae}.* By complete elimination the 2-form Σ^a as in [6], one also eliminates the flexibility of implementing the Hamiltonian and diffeomorphism constraints in (44). These are necessary for the construction of the metric $g_{\mu\nu}$, which plays the dual roles of solving the Einstein equations and enforcing Hodge duality. Additionally, in equation (2.8) in [6] the matrix Ψ does not have an antisymmetric part, wherereas $\Psi_{[ae]}$ was necessary in order to obtain (45) as well as the shift vector N^i. These two features constitute a vital part of the Hodge duality condition (51).

*The exception to this is the time gauge $e^0_i=0$, from which (18) follows. This has the effect of fixing the boost parameters corresponding to the local Lorentz frame. Since the SO(3,C) and SU(2) Lie algebras are isomorphic, (1) can be regarded as being based on the self-dual $SU(2)_-$ part of the Lorentz algebra, which leaves open the interpretation of the antiself dual part $SU(2)_+$. Since only $SU(2)_-$ is needed in order to obtain General Relativity, it could be that e^0_i is somehow associated with $SU(2)_+$. On a separate note, we have preserved the temporal 2-form components Σ^a_{0i} in I_{Inst}, in order to preserve the freedom to implement the initial value constraints.

The Einstein equations for $\Lambda = 0$ can be derived from (61), which is shown as equations (2.19a), (2.19b) and (2.20a) in [6]. But the statement that the metric (equations (2.2) and (2.4) in [6]) arises as a solution to these Einstein equations appears to the best of the present author's knowledge to be a separate postulate not derivable directly from (61). We will show explicitly in the present paper that this metric is the same one arising from the Hodge duality condition (45), and complete the missing link in this loop regarding the Einstein equations.

§7. The spacetime metric: revisited. We have shown that the instanton representation I_{Inst}, on-shell, implies a Hodge self-duality condition for the SO(3,C) curvature $F^a_{\mu\nu}$ with respect to a spacetime metric $g_{\mu\nu}$ solving the Einstein equations which also follow from I_{Inst}. All that is needed to construct the 3-metric h_{ij} for this spacetime metric are the spatial connection A^a_i and the CDJ matrix Ψ_{ae} solving the initial value constraints (44). The specification of the shift vector N^i via $A^a_0 \subset A^a_\mu = (A^a_0, A^a_i)$, combined with a lapse function N, then completes the construction of $g_{\mu\nu}$ via (40). We will see that I_{Inst} provides an additional simple formula for constructing $g_{\mu\nu}$ via the concept of Hodge duality. The Hodge self-duality condition (59) is given by

$$\epsilon_{ijk} B^j_a N^k + i\,\underline{N}\, h_{ij} B^j_a = -F^a_{0i}\,. \tag{67}$$

Multiplying (67) by $(B^{-1})^a_m$, we obtain the relation

$$\epsilon_{ijk} N^k + i\,\underline{N}\, h_{ij} = -F^a_{0i}\left(B^{-1}\right)^a_j . \tag{68}$$

Equation (68) provides a prescription for writing the spacetime metric explicitly in terms of the connection as follows.* The antisymmetric part of (68) yields the shift vector

$$N^k = -\frac{1}{2}\,\epsilon^{kij} F^a_{0i}\left(B^{-1}\right)^a_j , \tag{69}$$

and the symmetric part yields the 3-metric up to a conformal factor

$$i\,\underline{N}\, h_{ij} = -F^a_{0(i}\left(B^{-1}\right)^a_{j)} \equiv -\,c_{(ij)}\,, \tag{70}$$

where we have defined $c_{ij} = F^a_{0i}(B^{-1})^a_j$. The determinant of (70) yields

$$-\,i\,\frac{N^3}{\sqrt{h}} = -\det\|c_{(ij)}\| \equiv -\,c \;\longrightarrow\; i\,\underline{N} = \frac{c}{N^2}\,. \tag{71}$$

*In other words, the physical degrees of freedom from the initial value constraint contained in (44) become absorbed into the definition of the 3-metric h_{ij}.

Substituting this relation back into (70) enables us to solve for h_{ij}

$$h_{ij} = -\frac{N^2}{c}\, c_{(ij)}\,. \tag{72}$$

Let us define the following densitized object $\underset{\sim}{c}_{(ij)} = c^{-1} c_{(ij)}$. Then the line element (40) can also be written as*

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = -N^2\big(dt^2 + \underset{\sim}{c}_{(ij)}\, \omega^i \otimes \omega^j\big)\,, \tag{73}$$

where we have defined the one forms $\omega^i = dx^i - N^i dt$, with N^i given by equation (69). Starting from a spacetime of Lorentzian (Euclidean) signature for the lapse function N real (imaginary), we have obtained a line element (73). This implies the following consistency conditions

$$\left.\begin{array}{l} \underset{\sim}{c}_{ij} > 0 \;\longrightarrow\; N \text{ imaginary} \;\longrightarrow\; \text{Euclidean signature} \\ \underset{\sim}{c}_{ij} < 0 \;\longrightarrow\; N \text{ real} \;\longrightarrow\; \text{Lorentzian signature} \end{array}\right\}. \tag{74}$$

The result is that every connection A^a_μ with nondegenerate magnetic field B^i_a, combined with a lapse function N, determines a spacetime metric $g_{\mu\nu}$ of signature given by (74) solving the Einstein equations.

An elegant formula was constructed by Urbantke, which determines the metric with respect to which a given SU(2) Yang-Mills curvature, is self-dual in the spacetime sense. The formula is given by [16]

$$\sqrt{-g}\; g_{\mu\nu} = \frac{4}{3}\,\eta\, f_{abc} F^a_{\mu\rho} F^b_{\alpha\beta} F^c_{\sigma\nu}\, \epsilon^{\rho\alpha\beta\sigma}. \tag{75}$$

Since we are treating General Relativity in analogy with Yang-Mills theory, it is relevant to perform a 3+1 decomposition of (75). The result of this decomposition is given by†

$$g_{00} \propto \det\|F^a_{0i}\|\,, \quad g_{0k} \propto \epsilon_{klm}\,\big(F^{-1}\big)^{0l}_c B^m_c\,, \quad g_{ij} \propto F^a_{0(i}\,\big(B^{-1}\big)^a_{j)}\,. \tag{76}$$

Comparison of (76) with (69) and (70) reveals that on-shell, the instanton representation of Plebanski gravity reproduces the Urbantke metric purely from an action principle. When the spatial part of the Urbantke metric is built from variables solving the constraints (44), then the Urbantke metric also solves the Einstein equations by construction.

*More precisely, since (40) as defined by (44) forms a subset of the line element defined by (73), then the equality of (40) with (73) must be regarded as a consistency condition. Since (67) contains a velocity $\dot{A}^a_i$ and (44) does not, then the interpretation is that the equality between the line elements (40) and (73) must enforce the time evolution of initial data satisfying the initial value constraints (44).

†See Appendix B for the details of the derivation.

§7.1. Reality conditions. Since the connection A^a_μ is allowed to be complex, then the line element (73) in general allows for complex metrics $g_{\mu\nu}$. General Relativity should correspond to the restriction of this to real-valued metrics, which implies certain conditions on A^a_μ such that $\underset{\sim}{c}_{ij}$ be real-valued in (74). So the imposition of reality conditions requires that the undensitized matrix $c_{ij} = F^a_{0i}(B^{-1})^a_j$ be either real or pure imaginary, which leads to two cases

$$\left.\begin{array}{l} c_{(ij)} \text{ real } \longrightarrow \text{ Euclidean signature} \\ c_{(ij)} \text{ imaginary } \longrightarrow \text{ Lorentzian signature} \end{array}\right\}. \tag{77}$$

We will see that (77) places restrictions on the connection A^a_μ for a spacetime of fixed signature. For a general A^a_μ satisfying the reality conditions, there is apparently no constraint fixing the signature of the spatial part of the metric h_{ij}.*

The metric is clearly real if one is restricted to connections having a real curvature $F^a_{\mu\nu}$. When $F^a_{\mu\nu}$ is complex then we must impose reality conditions requiring $\underset{\sim}{c}_{ij}$ to be real as in (74). The symmetric part of this enforces reality of the 3-metric h_{ij} and the antisymmetric part enforces reality of the shift vector N^i. The lapse function N must always be chosen to be either real or pure imaginary. The signature of spacetime, which in either case apparently may change, might be more directly related to the reality of the metric. This is unlike the case in the Ashtekar variables, where for Euclidean signature spacetimes one is restricted to real variables.

We will now delineate the reality conditions on the spacetime metric for the case where the curvature $F^a_{\mu\nu}$ is complex. First let us perform the following split of the connection A^a_μ into the real and imaginary parts of its spatial and temporal components

$$A^a_i = \big(\Gamma - iK\big)^a_i\,, \qquad A^a_0 = \big(\eta - i\zeta\big)^a. \tag{78}$$

Corresponding to this 3+1 split, there is an analogous 3+1 split induced upon $F^a_{\mu\nu}$ into spatial and temporal components. The spatial part of this defines the magnetic field B^i_a given by

$$B^i_a = (R - iT)^i_a\,, \tag{79}$$

*Hence there is a caveat associated with the labels "Euclidean" and "Lorentzian" used in (77). The lapse function N is freely specifiable, since it is not constrained by A^a_μ. But it is still conceivable in (77) that different components of $\underset{\sim}{c}_{(ij)}$ could have different signs based on the initial data of A^a_μ. If this were to be the case, then this could bring in the possibility of topology changes for spacetimes described by I_{Inst} if the signature were not preserved under time-evolution.

where we have defined

$$\left.\begin{aligned} R^i_a &= \epsilon^{ijk}\,\partial_j\Gamma^a_k + \frac{1}{2}\,\epsilon^{ijk} f_{abc}\Gamma^b_j\Gamma^c_k - \frac{1}{2}\,\epsilon^{ijk}\,f^{abc}K^b_jK^c_k \\ T^i_a &= \epsilon^{ijk}\,\mathrm{D}_jK^a_k = \epsilon^{ijk}\big(\partial_jK^a_k + f^{abc}\Gamma^b_jK^c_k\big) \end{aligned}\right\}. \tag{80}$$

The quantity T^i_a is the covariant curl of K^a_i using Γ^a_i as a connection. The temporal part of the curvature $F^a_{\mu\nu}$ is given by

$$F^a_{0i} = \big(f - ig\big)^a_i\,, \tag{81}$$

where we have defined

$$\left.\begin{aligned} f^a_i &= \dot{\Gamma}^a_i - \mathrm{D}_i\,\eta^a + f^{abc}K^b_i\zeta^c \\ g^a_i &= \mathrm{D}_0\,K^a_i - \mathrm{D}_i\,\zeta^a \end{aligned}\right\}. \tag{82}$$

The operator D_i is the covariant derivative with respect to Γ^a_i as in the second line of (80), and D_0 is given by

$$\mathrm{D}_0K^a_i = \dot{K}^a_i + f^{abc}\eta^bK^c_i\,. \tag{83}$$

For the general complex case, reality conditions require that $c_{ij} = (B^{-1})^a_i F^a_{0j}$ be either real or pure imaginary as in (77). It will be convenient to use the following matrix identity, suppressing the indices

$$B^{-1} = (R - iT)^{-1} = (1 + iRT)\big[1 - (R^{-1}T)^2\big]^{-1}R^{-1}, \tag{84}$$

which splits the inverse of a complex matrix into its real and imaginary parts. Then upon contraction of the internal indices, c_{ij} is given by

$$\begin{aligned}(f - ig)(R - iT)^{-1} = \big[\,f + gR^{-1}T + i(-g + fR^{-1}T)\big] \times \\ \times\big[\,1 - (R^{-1}T)^2\big]R^{-1}.\end{aligned} \tag{85}$$

The last two matrices in (85) are real and the first matrix is in general complex. For Lorentzian signature spacetimes we must require the real part of the first matrix to be zero, and for Euclidean signature we must require the imaginary part to be zero. This leads to the matrix equations

$$\left.\begin{aligned} &\text{Euclidean signature:} && g^{-1}f = -R^{-1}T \\ &\text{Lorentzian signature:} && f^{-1}g = R^{-1}T \end{aligned}\right\}. \tag{86}$$

The aforementioned caveats still apply with respect to the stability of the signature. But in either case the reality conditions constitute

9 equations in 24 unknowns, namely the 12 complex components of the four-dimensional SO(3,C) connection A^a_μ. After implementation of these reality conditions, then this leaves $24-9=15$ real degrees of freedom in A^a_μ.*

§8. Gravity as a "generalized" Yang-Mills theory. We will now show how the concept of Hodge self-duality stems at a more fundamental level from internal duality with respect to gravitational degrees of freedom. Let us start off by considering the following action which resembles SO(3,C) Yang-Mills theory in curved spacetime

$$I = \int_{\mathrm{M}} d^4x \left(-\frac{1}{4}\sqrt{-g}\; g^{\mu\nu} g^{\rho\sigma} F^b_{\mu\nu} F^f_{\rho\sigma} \Psi_{bf} + \frac{1}{G}\sqrt{-g}\, R \right), \qquad (87)$$

where $R=R[g]$ is meant to signify that R is the curvature of the same metric which appears in the Yang-Mills term.

The quantity $g^{\mu\nu}$ is the covariant metric corresponding to the background spacetime upon which a Yang-Mills field A^a_μ propagates, and $F^a_{\mu\nu}$ is the curvature of A^a_μ, given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \,, \qquad (88)$$

where $f^{abc}=\epsilon^{abc}$ are the structure constants of SO(3,C).

Equation (87) is different from the usual Yang-Mills theory in that the two curvatures $F^a_{\mu\nu}$ additionally couple to a field Ψ_{bf} taking its values in two copies of SO(3,C). In the special case $\Psi_{ae}=k\,\delta_{ae}$ for some numerical constant k, Ψ_{ae} plays the role of the Cartan-Killing metric for the SO(3,C) Lie algebra. There is a wide array of literature concerning gravity and Yang-Mills theory, where one attempts to solve (87) for the Yang-Mills field A^a_μ as well as for the metric $g_{\mu\nu}$. But in the gravitational context, $\Psi_{ae}=-\frac{3}{\Lambda}\,\delta_{ae}$ implies that the metric $g_{\mu\nu}$ must be restricted to spacetimes of Petrov Type O, since Ψ_{ae} then has three equal eigenvalues [10].

The implication is that when one solves (87) in the case $\Psi_{ae}=-\frac{3}{\Lambda}\,\delta_{ae}$, then one is solving the coupled Yang-Mills theory only for conformally flat spacetimes. But we would like to incorporate more general geometries. On the one hand in vacuum Yang-Mills theory one already has a Yang-Mills solution for known metrics by virtue of Hodge duality and the Bianchi identity. On the other hand, the generalization of Ψ_{ae} to

*For example in (78), then one possibility is to regard the nine components of $\mathfrak{Re}\{A^a_i\}$ as freely specifiable, and then use (86) to determine the nine components of $\mathfrak{Im}\{A^a_i\}$ in terms of them.

include gravitational degrees of freedom, as we will see, enables one to identify the Yangs-Mills theory with the gravity theory that it is coupling to. To see this, let us split the Yang-Mills part of the Lagrangian of (87) into its spatial and temporal parts

$$L_{\mathrm{YM}} = \frac{\sqrt{-g}}{2}\Big(g^{00}g^{ij}F^b_{0i}F^f_{0j} - g^{0i}g^{0j}F^b_{0i}F^f_{0j} + \\ + 2g^{0i}g^{jk}F^b_{ij}F^f_{0k} + \frac{1}{2}\,g^{ik}g^{jl}F^b_{ij}F^f_{kl}\Big)\,\Psi_{bf}\,, \qquad (89)$$

where $F^a_{0i} = \dot{A}^a_i - \mathrm{D}_i A^a_0$ is the temporal component of the curvature.

The electric field is the momentum canonically conjugate to the Yang-Mills spatial connection

$$\Pi^i_b = \frac{\delta I_{\mathrm{YM}}}{\delta \dot{A}^b_i} = \sqrt{-g}\,\Big(g^{00}g^{ij}F^f_{0j} - g^{0i}g^{0j}F^f_{0j} + g^{0m}g^{ni}F^f_{mn}\Big)\,\Psi_{bf}\,. \qquad (90)$$

Next, we will make use of the $3+1$ decomposition of the spacetime metric

$$g^{\mu\nu} = \begin{pmatrix} g^{00} & g^{0i} \\ g^{0j} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{N^2} & -\frac{N^i}{N^2} \\ -\frac{N^j}{N^2} & h^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix},$$

where $N^\mu = (N, N^i)$ are the lapse function and shift vector, and $\sqrt{-g} = N\sqrt{h}$ is the determinant of $g_{\mu\nu}$. Substitution into (90) yields

$$\Pi^i_b = \frac{\sqrt{h}}{N}\,\Big(-h^{ij}F^f_{0j} + N^m h^{ni}F^f_{mn}\Big)\,\Psi_{bf}\,, \qquad (91)$$

and substitution into (89) yields

$$L_{\mathrm{YM}} = -\frac{1}{2}\,N\sqrt{h}\,\Big[-\frac{1}{N^2}\,h^{ij}F^b_{0i}F^f_{0j} + 2\,\frac{N^i}{N^2}\Big(h^{jk} - \frac{N^j N^k}{N^2}\Big)F^b_{ij}F^f_{0k} + \\ + \frac{1}{2}\,h^{ik}\Big(h^{jl} - \frac{2N^j N^l}{N^2}\Big)F^b_{ij}F^f_{kl}\Big]\Psi_{bf}\,. \qquad (92)$$

We will now eliminate the velocities $\dot{A}^a_i$ from (92) by inverting (91)

$$F^f_{0j} = h_{jk}\,\Big[-\frac{N}{\sqrt{h}}\,\Pi^k_b\,(\Psi^{-1})^{bf} + N^m h^{nk}F^f_{mn}\Big]\,. \qquad (93)$$

Upon substitution of (93) into (92) after several long but straightforward algebraic steps, we obtain

$$L_{\mathrm{YM}} = \frac{1}{2}\,\frac{N}{\sqrt{h}}\,h_{ij}\,\Pi^i_b\,\Pi^j_f\,(\Psi^{-1})^{bf} + \frac{1}{4}\,N\sqrt{h}\,h^{ik}h^{jl}F^b_{ij}F^f_{kl}\Psi_{bf}\,. \qquad (94)$$

Defining the SO(3,C) magnetic field by $F^a_{ij} = \epsilon_{ijk} B^k_a$, and using the relation

$$\frac{1}{2}\,\epsilon_{ijm}\,\epsilon_{kln} h^{ik} h^{jl} = \frac{1}{h}\,h_{mn}\,, \tag{95}$$

and presupposing the 3-metric h_{ij} to be nondegenerate, then (94) yields

$$L_{\rm YM} = \frac{1}{2}\,\underline{N}\,h_{ij}\left[(\Psi^{-1})^{bf}\,\Pi^i_b \Pi^j_f - \Psi_{bf} B^i_b B^j_f\right]. \tag{96}$$

This is the electromagnetic decomposition of the generalized Yang–Mills action, with Ψ_{bf} replacing the invariant Cartan-Killing form for the SO(3) gauge group. But for geometries not of Petrov Type O, then Ψ_{bf} is in general no longer SO(3,C) invariant.

To see how General Relativity follows from this "generalized" Yang-Mills theory, let us impose the following relation between the electric and the magnetic fields of the latter

$$\Pi^i_a = \beta\,\Psi_{ae} B^i_e \tag{97}$$

for some numerical constant β. Then for nondegenerate Ψ_{bf}, substitution of (97) into (90) implies that

$$\beta B^i_f = N\sqrt{h}\left(g^{00} g^{ij} F^f_{0j} - g^{0i} g^{0j} F^f_{0j} + g^{0m} g^{ni} F^f_{mn}\right). \tag{98}$$

The right hand side of (98) is given by

$$N\sqrt{h}\left[-\frac{1}{N^2}\left(h^{ij} - \frac{N^i N^j}{N^2}\right) F^f_{0j} - \frac{N^i N^j}{N^4}\,F^f_{0j} - {} \right. \\ \left. - \frac{N^m}{N^2}\left(h^{ni} - \frac{N^n N^i}{N^2}\right) F^f_{mn}\right], \tag{99}$$

which simplifies to

$$\frac{\sqrt{h}}{N}\left(h^{ij} F^f_{0j} + N^k h^{ij} F^f_{kj}\right) = -\beta B^i_f\,. \tag{100}$$

Equation (100) can be rewritten as

$$F^f_{0j} + \epsilon_{jmk} B^m_f N^k + \beta\,\underline{N}\,h_{ji} B^i_f = 0\,. \tag{101}$$

The choice $\beta = \pm i$ would imply that equation (97) automatically imposes Hodge self-duality of the Yang-Mills curvature $F^f_{\mu\nu}$ with respect to the metric $g_{\mu\nu}$ which it couples to, namely

$$H^{\mu\nu\rho\sigma} F^b_{\rho\sigma} = 0\,, \tag{102}$$

where we have defined the Hodge self-duality operator

$$H^{\mu\nu\rho\sigma} = \frac{1}{2}\big[\sqrt{-g}\,(g^{\mu\rho}g^{\nu\sigma} - g^{\nu\rho}g^{\mu\sigma}) + \beta\,\epsilon^{\mu\nu\rho\sigma}\big]. \tag{103}$$

Comparison of (97) with the spatial restriction of equation the third equation of (2), and comparison of (101) with (51), implies that (97) is the internal analogue of Hodge self-duality. Indeed, the fact that the metric defining (102) solves the Einstein equations transforms (34) on-shell into (87). Since the solutions to ordinary vacuum Yang-Mills theory include Yang-Mills instantons, then this suggests that I_{Inst} is a theory which should include gravitational instantons.

§9. Gravitational instantons: revisited. We will now put into context the points raised in the introduction paragraph regarding the apparent ambiguity in the definition of gravitational instantons. It has been noted by Ashtekar and Renteln [1] that the ansatz

$$B^i_a = -\frac{\Lambda}{3}\,\widetilde{\sigma}^i_a\,, \tag{104}$$

solves the initial value constraints of the Ashtekar variables arising from (13). It was noted that this corresponds to the conformally flat spacetimes.* There is a covariant form of the action (13) provided by Samuel [18, 19] in which the basic variables are two forms $\Sigma^b = \frac{1}{2}\,\Sigma^b_{\mu\nu}dx^\mu\wedge dx^\nu$, given by

$$I = \int_{\mathrm{M}} d^4x\left(\Sigma^b_{\mu\nu}F^b_{\rho\sigma} + \frac{\Lambda}{6}\,\Sigma^b_{\mu\nu}\Sigma^b_{\rho\sigma}\right)\epsilon^{\mu\nu\rho\sigma}. \tag{105}$$

Equation (105) leads to General Relativity with cosmological constant through the equations of motion

$$\epsilon^{\mu\nu\rho\sigma}\,\mathrm{D}_\nu\Sigma^b_{\rho\sigma} = 0\,, \qquad F^b_{\mu\nu} = -\frac{\Lambda}{3}\,\Sigma^b_{\mu\nu}\,, \tag{106}$$

where the two form is constructed from SL(2,C) one forms

$$\Sigma^{AB}_{\mu\nu} = i\left(e^{AA'}_\mu e^B_{\nu A'} - e^{AA'}_\nu e^B_{\mu A'}\right) \tag{107}$$

in self-dual combination. The class of solutions described by the second equation of (106) are the evolution of (104), which is the data set on the initial spatial hypersurface. The observation that the first equation

*We will see that (97) is the generalization of (104) which incorporates more general geometries including Types D and O, when Ψ_{ae} becomes identified with the CDS matrix Ψ_{ae} of I_{Inst}.

of (106) follows identically from the second due to the Bianchi identity, combined with the self duality in (107) allows an association of gravitation with Yang-Mills instantons to be inferred [18].

It was postulated that there might be other Yang-Mills field strengths which satisfy (106), but one is limited to conformally flat metrics since not all two forms Σ^a are constructible from tetrad one forms $e^{AA'}_{\mu}$ as in (107). The problem of relating (106) to the Yang-Mills self-duality condition ${}^*F = F$ resides in the observation that the metric $g_{\mu\nu}$ must first be known. In [7], Jacobson eliminates the tetrad from the self-duality condition to address the sector with vanishing self-dual Weyl curvature, by proposing the following condition on the curvature

$$F^b \wedge F^f - \frac{1}{3}\,\delta^{bf}\,\mathrm{tr}\,F \wedge F = 0\,. \tag{108}$$

Given a connection A^a_μ which solves (108), the tetrads in (107) associated with the 2-forms Σ^b determine a metric which is a self-dual Einstein solution with cosmological constant Λ. Moreover, the curvature satisfying (108) is self-dual with respect to this metric. Since (108) is the same as the second equation of (2) when $\Psi_{ae} \propto \delta_{ae}$, then the problem of "finding the metric" as pointed out by Samuel in [18] translates into the problem of finding the connection in (108).

Hence the aforementioned developments have been shown only for the conformally self-dual case where the self-dual Weyl tensor ψ_{ae} vanishes, whence the metric is explicitly constructible. This limits one to spacetimes of Petrov Type O.* The proposition of the present paper has been to extend the library of solutions to include the Petrov Types I and D cases using I_{Inst}.

§9.1. Generalization beyond Petrov Type O instantons. We have seen that the CDJ ansatz, the spatial restriction of the third equation of (2), imposes the condition of Hodge self-duality on the "generalized" SO(3,C) Yang-Mills fields in (97). When Ψ_{ae} is chosen to satisfy the constraints (44), then the implication is that this Yang-Mills theory becomes a theory of General Relativity. Since vacuum Yang-Mills theory in conformally flat spacetimes describes instantons, then this suggests that the gravitational analogue of pure Yang-Mills theory must describe gravitational instantons, specifically incorporating the physical

*In [8] gravitational instantons are defined as spacetimes with vanishing self-dual Weyl curvature, and nonvanishing cosmological constant. This falls within the Petrov Type O case with $\Psi_{ae} = -\frac{3}{\Lambda}\,\delta_{ae}$, with no restrictions on the connection A^a_i. We would like to generalize this to incorporate Type D and Type I spacetimes.

degrees of freedom from (44). To examine the implications for gravity let us recount the action (34), repeated here for completeness

$$I_{\text{Inst}} = \int_{\text{M}} d^4x \, \frac{1}{8} \, \Psi_{ae} F^a_{\mu\nu} F^e_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} - i\sqrt{-g} \left(\Lambda + \text{tr}\,\Psi^{-1}\right), \qquad (109)$$

which corresponds to (19) at the level after elimination of $\Psi_{[ae]}$ and the shift vector N^i.

Recall also that the equation of motion for Ψ_{ae} prior to elimination of N^i and in (45) implies the Hodge self-duality condition

$$\beta \, \epsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma} = \sqrt{-g} \, g^{\mu\rho} g^{\nu\sigma} F^a_{\rho\sigma} \qquad (110)$$

once one has made the identification of $h_{ij} = h_{ij}[\Psi, A]$. Substitution of (110) into the first term of (109) yields

$$I_{\text{Inst}} = \int_{\text{M}} d^4x \, \frac{\beta}{4} \sqrt{-g} \, g^{\mu\rho} g^{\nu\sigma} F^a_{\mu\nu} F^e_{\rho\sigma} \Psi_{ae} - i\sqrt{-g} \left(\Lambda + \text{tr}\,\Psi^{-1}\right), \quad (111)$$

which is nothing other than the action for gravity coupled to a "generalized" SO(3,C) Yang-Mills theory of gravity (87). On the other hand, the equation of motion for Ψ_{ae} derived from (109) is

$$\frac{1}{8} F^b_{\mu\nu} F^f_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} + i\sqrt{-g} \left(\Psi^{-1}\Psi^{-1}\right)^{fb} = 0 \,. \qquad (112)$$

Comparison of (112) with (43) indicates that dynamically on the solution to the equations of motion,

$$\frac{1}{8} F^b_{\mu\nu} F^f_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} = -i\beta^{-1/2} N \left(\det\|B\|\right)^{-1/2} \left(\det\|\Psi\|\right)^{-1/2} \times$$
$$\times \, h_{ij} B^i_b B^j_f = -i\beta \, \underline{N} h_{ij} B^i_b B^j_f \,, \qquad (113)$$

where $\underline{N} = N h^{-1/2}$. Since the initial value constraints must be consistent with the equations of motion we can insert (113) into (109), which yields

$$I_{\text{Inst}} = \frac{\beta}{2} \int_{\text{M}} \Psi_{ae} F^a \wedge F^e = -i\beta \int_{\text{M}} \underline{N} h_{ij} \, \Psi_{ae} B^i_a B^j_e \, d^4x \,. \qquad (114)$$

But equation (114) is only the magnetic part of a Yang-Mills theory in curved spacetime. To obtain the respective electric part we use the relation $B^i_e = \frac{1}{\beta} \Psi^{-1}_{ae} \widetilde{\sigma}^i_a$, which shows on-shell that the following objects are equivalent

$$-\, i\beta \underline{N} h_{ij} B^i_b B^j_f \Psi_{bf} = -i \underline{N} h_{ij} \, \widetilde{\sigma}^i_b B^j_f =$$
$$= -i\beta \underline{N} h_{ij} \left(\Psi^{-1}\right)^{bf} \widetilde{\sigma}^i_b \widetilde{\sigma}^j_f \,. \qquad (115)$$

So we can use (115) to eliminate B^i_a from (114), yielding

$$I_{\text{Inst}} = \frac{\beta}{2}\int_{\text{M}} \Psi_{ae} F^a \wedge F^e = -i\beta \int_{\text{M}} \frac{1}{\beta^2}\, \underline{N}\, h_{ij}\, (\Psi^{-1})^{ea}\, \widetilde{\sigma}^i_a \widetilde{\sigma}^j_e\, d^4x\,. \tag{116}$$

The action for the instanton representation I_{Inst} evaluated on a classical solution can be written as the average of the actions (114) and (116), which yields

$$I_{\text{Inst}} = \frac{i\beta}{2}\int dt \int_{\Sigma} d^3x\, \underline{N}\, h_{ij}\left[-\frac{1}{\beta^2}\,(\Psi^{-1})^{bf}\,\widetilde{\sigma}^i_b \widetilde{\sigma}^j_f - \Psi_{bf} B^i_b B^j_f\right] =$$

$$= i\beta \int dt \int_{\Sigma} d^3x \left[\underline{N}\, h_{ij} T^{ij} - \frac{i}{2}\,\beta\left(1 + \frac{1}{\beta^2}\right)\underline{N}\, h_{ij}\, \widetilde{\sigma}^i_b \widetilde{\sigma}^j_f\, (\Psi^{-1})^{bf}\right] \tag{117}$$

with T^{ij} given by

$$T^{ij} = \frac{1}{2}\left[(\Psi^{-1})^{ae}\,\widetilde{\sigma}^i_a \widetilde{\sigma}^j_e - \Psi_{ae} B^i_a B^j_e\right]. \tag{118}$$

With the exception of the term proportional to β, (117) would be the action for a "generalized" Yang-Mills theory. Note that it is a genuine Yang-Mills theory only for $\Psi_{ae} = k\,\delta_{ae}$, which covers only the Type O sector of gravity.

Upon making the identification $\widetilde{\sigma}^i_a \equiv \Pi^i_a$ from (96), then we have on the solution to the equations of motion that

$$\frac{1}{8}\int_{\text{M}} d^4x\, \Psi_{bf} F^b_{\mu\nu} F^f_{\rho\sigma}\, \epsilon^{\mu\nu\rho\sigma} = i\beta \int_{\text{M}} d^4x\, \sqrt{-g}\, g^{\mu\rho} g^{\nu\sigma} F^b_{\mu\nu} F^f_{\rho\sigma} + Q\,, \tag{119}$$

where Q is the second term in the bottom line of (117). The identification between the Yang-Mills and the instanton representation actions can be made only for $\beta^2 = -1$. In this case $Q = 0$ and equation (119) implies on the solution to the equations of motion that

$$\frac{1}{8}\int_{\text{M}} d^4x \left(\sqrt{-g}\, g^{\mu\rho} g^{\nu\sigma} - g^{\nu\rho} g^{\mu\sigma} \pm i\,\epsilon^{\mu\nu\rho\sigma}\right) F^b_{\mu\nu} F^f_{\rho\sigma} \Psi_{bf} = 0\,. \tag{120}$$

In order for this to be true for all curvatures, we must have

$$\pm \frac{i}{2}\,\epsilon^{\mu\nu\rho\sigma} F^f_{\rho\sigma} = \sqrt{-g}\, g^{\mu\rho} g^{\nu\sigma} F^f_{\rho\sigma}\,, \tag{121}$$

namely that the curvature of the starting theory must be self-dual in the Hodge sense in any solution to the equations of motion. In this case, it can be said that General Relativity is literally a Yang-Mills theory coupled gravitationally to itself.

§10. Summary. In this paper we have presented the instanton representation of Plebanski gravity, a new formulation of General Relativity. The basic dynamical variables are an SO(3,C) gauge connection A^a_μ and a matrix Ψ_{ae} taking its values in two copies of SO(3,C). The consequences of the associated action I_{Inst} were determined via its equations of motion with the following results:

1) The two equations of motion for I_{Inst} imply the Einstein equations when the initial value constraints are satisfied;
2) When these constraints are satisfied, then one can define a spatial 3-metric $h_{ij}[\Psi, A]$ using Ψ_{ae} and A^a_i, the spatial part of the connection A^a_μ;
3) The first equation of motion for I_{Inst} is consistent with the second equation when the initial value constraints are satisfied;
4) The first equation of motion of I_{Inst} implies that the curvature $F^a_{\mu\nu}$ is Hodge self-dual with respect to the metric $g_{\mu\nu}$ which solves the Einstein equations as a consequence of the initial value constraints.

Each of these results hinges crucially on the existence of solutions to the initial value constraints. So it remains to be verified that that once the initial value constraints are satisfied on an initial spatial hypersurface, then the equations of motion should preserve these constraints for all time. We will relegate demonstration of this for a future publication.

Additionally, we have clarified the relation between I_{Inst} and I_{CDJ} in [5]. The two formulations are not the same as it may naively appear for the following reasons:

1) The action I_{CDJ} at the level prior to elimination of Ψ_{ae} from I_{Pleb} is missing the 2-forms $\Sigma^a_{\mu\nu}$ as well as the antisymmetric part of Ψ_{ae}. However, I_{Inst} contains Σ^a_{0i}, the temporal part of $\Sigma^a_{\mu\nu}$ as well as $\Psi_{[ae]}$;
2) The Hodge duality condition follows directly as an equation of motion for I_{Inst}, a crucial part of which involves $N^\mu = (N, N^i)$ from Σ^a_{0i} which are needed both for constructing General Relativity solutions as well as for implementing the initial value constraints*;
3) The reality conditions in I_{Inst} appear to be intimately connected with the signature of spacetime as well as initial data, which is unlike the usual formulations of General Relativity. The implications of this should be borne out when one attempts to construct solutions.

*The advantages of these features should become more apparent when one proceeds to construct General Relativity solutions and in the quantum theory.

The instanton representation $I_{\rm Inst}$ has exposed an interesting relation between General Relativity and Yang-Mills theory, which suggests that this is indeed a theory of "generalized" Yang-Mills instantons. In the conformally flat case, the CDJ matrix Ψ_{ae} has three equal eigenvalues and thus plays the role of a Cartan-Killing SO(3,C) invariant metric. The generalization of this to more general geometries presents an interesting physical interpretation, since Ψ_{ae} contains gravitational degrees of freedom. In the Petrov Type D case for example, where Ψ_{ae} has two equal eigenvalues, then the Yang-Mills SO(3) symmetry becomes broken down to SO(3,C). In the algebraically general Type I case, where $\lambda_1 \neq \lambda_2 \neq \lambda_3$, the SO(3,C) symmetry becomes completely broken. A possible future direction is to investigate possible mechanisms which could induce such a breaking of this symmetry.

Nevertheless, the first order of business in future research will be to check for consistency of the initial value constraints of $I_{\rm Inst}$ under time evolution. Then next will be to use $I_{\rm Inst}$ reconstruct as many of the known General Relativity solutions as possible and to construct new solutions. Additionally, we will examine the quantum theory with a view to addressing many of the unresolved questions in quantum gravity.

§10.1. Preview into the quantum theory. Instantons in Yang-Mills theory can be associated with transitions between topologically inequivalent vacua, induced by tunnelling classical solutions upon Wick rotation between Lorentzian and Euclidean signature spacetimes. A future direction of research will be to investigate the analogue of this feature for $I_{\rm Inst}$, in addition to the quantum aspects of the theory. For instance, upon substitution of contraction of (112) with Ψ_{bf} one obtains the relation

$$\frac{1}{8}\,\Psi_{bf}\,F^b_{\mu\nu}\,F^f_{\rho\sigma}\,\epsilon^{\mu\nu\rho\sigma} = -\,i\sqrt{-g}\;{\rm tr}\,\Psi^{-1} = i\sqrt{-g}\,\Lambda\,, \tag{122}$$

where we have used the Hamiltonian constraint from variation of N in (109). Substitution of (122) back into (109) yields

$$I_{\rm Inst} = i\,\Lambda\int_{\rm M} d^4x\,\sqrt{-g} = i\,\Lambda\,{\rm Vol(M)}\,, \tag{123}$$

where Vol(M) is the spacetime volume. The exponentiation of this in units of $\hbar G$ yields

$$\psi = e^{i\Lambda(\hbar G)^{-1}{\rm Vol(M)}}, \tag{124}$$

which forms the dominant contribution to the path integral for gravity due to gravitational instantons [20]. On the other hand, substitution of

$\Psi_{ae}=-\frac{3}{\Lambda}\delta_{ae}$ into the starting action (19) produces a total derivative leading via Stokes' theorem to a Chern-Simons boundary term I_{CS}. The exponentiation of this boundary term in units of $\hbar G$ yields

$$\psi_{\text{Kod}}=e^{\pm 3(2\hbar G\Lambda)^{-1}\int_{\text{M}}\text{tr}F\wedge F}=e^{\pm 3(\hbar G\Lambda)^{-1}I_{\text{CS}}[A]}, \tag{125}$$

which is known as the Kodama state which describes de Sitter spacetime [21, 22]. One of the results of the quantum theory of I_{Inst} should be to clarify the role of (125) in quantum gravity, and to attempt to find its counterparts for Ψ_{ae} corresponding to more general spacetime geometries. The generalization of the left hand side of (125) is

$$\psi_{\text{Inst}}=e^{(2\hbar G)^{-1}\int_{\text{M}}\Psi_{ae}F^a\wedge F^e}. \tag{126}$$

As part of the investigation of the quantum theory one would like to find the analogue of the right hand side of (125) for (126).

Appendix A. Components of the Hodge self-duality operator. From the equation

$$N\sqrt{h}\Big[\big(g^{00}g^{kj}-g^{k0}g^{0j}\big)F^a_{0j}+g^{0i}g^{kj}\epsilon_{ijm}B^m_a\Big]=\beta B^k_a \tag{127}$$

from (55), we have

$$N\sqrt{h}\left\{\left[-\frac{1}{N^2}\left(h^{kj}-\frac{N^kN^j}{N^2}\right)-\left(\frac{N^kN^j}{N^2}\right)\right]F^a_{0j}-\right.$$
$$\left.-\frac{N^i}{N^2}\left(h^{kj}-\frac{N^kN^j}{N^2}\right)\epsilon_{ijm}B^m_a\right\}=\beta B^k_a. \tag{128}$$

Cancelling off the terms multipying F^a_{0j} which are quadratic in N^i, we have

$$-\frac{\sqrt{h}}{N}h^{kj}\big(F^a_{0j}+\epsilon_{jmi}B^m_aN^i\big)=\beta B^k_a. \tag{129}$$

Multiplying (129) by $\underline{N}=Nh^{-1/2}$ and by h_{lk}, this yields

$$F^a_{0l}+\epsilon_{lmi}B^m_aN^i+\beta\underline{N}h_{lk}B^k_a=0. \tag{130}$$

From the equation

$$N\sqrt{h}\Big[\big(g^{m0}g^{nj}-g^{n0}g^{mj}\big)F^a_{0j}+g^{mi}g^{nj}\epsilon_{ijk}B^k_a\Big]=\beta\epsilon^{mn0j}F^a_{0j}, \tag{131}$$

from (57), we have

$$N\sqrt{h}\left\{\left[-\frac{N^m}{N^2}\left(h^{nj}-\frac{N^nN^j}{N^2}\right)+\frac{N^n}{N^2}\left(h^{mj}-\frac{N^mN^j}{N^2}\right)\right]F^a_{0j}+\right.$$
$$\left.+\left(h^{mi}-\frac{N^mN^i}{N^2}\right)\left(h^{nj}-\frac{N^nN^j}{N^2}\right)\epsilon_{ijk}B^k_a\right\}=\beta\,\epsilon^{0mnj}F^a_{0j}\,. \quad (132)$$

Expanding and using the vanishing of the term quadratic in the shift vector N^i, we have

$$\frac{\sqrt{h}}{N}\Big\{\big(h^{mj}N^n-h^{nj}N^m\big)F^a_{0j}+N\sqrt{h}\,h^{mi}h^{nj}\epsilon_{ijk}B^k_a-$$
$$-\big(h^{mi}N^nN^j+h^{nj}N^mN^i\big)\,\epsilon_{ijk}B^k_a\Big\}=\beta\,\epsilon^{0mnj}F^a_{0j}\,. \quad (133)$$

From the third term on the left hand side of (133), we have the following relation upon relabelling indices $i\leftrightarrow j$ on the first term in brackets

$$-h^{mi}N^nN^j\epsilon_{ijk}B^k_a-h^{nj}N^mN^i\epsilon_{ijk}B^k_a=-h^{mj}N^nN^i\epsilon_{jik}B^k_a-$$
$$-h^{nj}N^mN^i\epsilon_{ijk}B^k_a=\epsilon_{ijk}\big(h^{mj}N^n-h^{nj}N^m\big)N^iB^k_a\,. \quad (134)$$

Note that the combination $h^{mj}N^n-h^{nj}N^m$ on the right hand side of (134) is the same term multiplying F^a_{0j} in the left hand side of (133). Using this fact, then (133) can be written as

$$\frac{\sqrt{h}}{N}\Big[\big(h^{mj}N^n-h^{nj}N^m\big)\big(F^a_{0j}+\epsilon_{jki}B^k_aN^i\big)\Big]+$$
$$+\underline{N}\,\epsilon^{mnl}h_{lk}B^k_a=\beta\,\epsilon^{mnj}F^a_{0j}\,, \quad (135)$$

where $\epsilon^{0mnj}=\epsilon^{mnj}$. Using $F^a_{0j}+\epsilon_{jki}B^k_aN^i=-\beta\underline{N}h_{jk}B^k_a$ from (130) in (135), then we have

$$-\frac{\sqrt{h}}{N}\big(h^{mj}N^n-h^{nj}N^m\big)\beta\underline{N}\,h_{jk}B^k_a+\underline{N}\,\epsilon^{mnl}h_{lk}B^k_a=\beta\,\epsilon^{mnj}F^a_{0j}\,. \quad (136)$$

This simplifies to

$$-\beta\big(\delta^m_kN^n-\delta^n_kN^m\big)B^k_a+\underline{N}\,\epsilon^{mnl}h_{lk}B^k_a=\beta\,\epsilon^{mnj}F^a_{0j}\;\longrightarrow$$
$$\longrightarrow\;\beta\big(\epsilon^{mnj}F^a_{0j}+B^m_aN^n-B^n_aN^m\big)=\underline{N}\,\epsilon^{mnj}h_{jk}B^k_a\,. \quad (137)$$

Contracting (137) with ϵ_{mnl} and dividing by 2, we obtain the relation

$$F^a_{0l}+\epsilon_{lmn}B^m_aN^n-\frac{1}{\beta}\,\underline{N}\,h_{lk}B^k_a=0\,. \quad (138)$$

Consistency of (138) with (130) implies that $\beta^2=-1$, or that $\beta=\pm i$.

Appendix B. Urbantke metric components. We now perform a 3+1 decomposition of the Urbantke metric*

$$g_{\mu\nu} = f_{abc} F^a_{\mu\rho} F^b_{\alpha\beta} F^c_{\sigma\nu} \epsilon^{\rho\alpha\beta\sigma}. \tag{139}$$

In what follows we define $\epsilon^{0123}=1$, and make use of the fact that the structure constants $f_{abc}=\epsilon_{abc}$ for SO(3,C) are numerically the same as the three-dimensional epsilon symbol. Also, we will use the definition $B^i_a=\frac{1}{2}\epsilon^{ijk}F^a_{jk}$ of the Ashtekar magnetic field. The main result of this appendix is that due to the symmetries of the four-dimensional epsilon tensor, each term in the expansion is the same to within a numerical constant. We will show this by explicit calculation.

1. Starting from the time-time component we have

$$g_{00} = f_{abc} F^a_{0\rho} F^b_{\alpha\beta} F^c_{\sigma 0} \epsilon^{\rho\alpha\beta\sigma}. \tag{140}$$

The time-time component of $g_{\mu\nu}$ reduces from two terms to one term

$$f_{abc} F^a_{0i} F^b_{0j} F^c_{k0} \epsilon^{i0jk} + f_{abc} F^a_{0i} F^b_{j0} F^c_{k0} \epsilon^{ijk0} = \\ = 2 f_{abc} \epsilon^{ijk} F^a_{0i} F^b_{0j} F^c_{0k} = 12 \det\|F^a_{0i}\|. \tag{141}$$

2. Moving on to the space-time components, we have

$$\begin{aligned} g_{0k} &= f_{abc} F^a_{0\rho} F^b_{\alpha\beta} F^c_{\sigma k} \epsilon^{\rho\alpha\beta\sigma} = f_{abc} F^a_{0i} F^b_{\alpha\beta} F^c_{\sigma k} \epsilon^{i\alpha\beta\sigma} = \\ &= f_{abc} F^a_{0i} F^b_{0j} F^c_{lk} \epsilon^{i0jl} + f_{abc} F^a_{0i} F^b_{j0} F^c_{lk} \epsilon^{ij0l} + f_{abc} F^a_{0i} F^b_{jl} F^c_{0k} \epsilon^{ijl0} = \\ &= -2 f_{abc} \epsilon^{ijl} \epsilon_{lkm} B^m_c F^a_{0i} F^b_{0j} - 2 f_{abc} F^a_{0i} F^c_{0k} B^i_b = \\ &= -2 f_{abc} \left(\delta^i_k \delta^j_m - \delta^i_m \delta^j_k\right) F^a_{0i} F^b_{0j} B^m_c - 2 f_{abc} F^a_{0i} F^c_{0k} B^i_b = \\ &= 2 f_{abc} F^a_{0m} F^b_{0k} B^m_c = 2 \left(\det\|F^a_{0i}\|\right) \epsilon_{mkl} \left(F^{-1}\right)^c_{0l} B^m_c. \end{aligned} \tag{142}$$

3. The spatial components are given by

$$g_{ij} = f_{abc} F^a_{i\rho} F^b_{\alpha\beta} F^c_{\sigma j} \epsilon^{\rho\alpha\beta\sigma} \tag{143}$$

which decomposes into a sum of four terms

$$g_{ij} = f_{abc} F^a_{i0} F^b_{mk} F^c_{lj} \epsilon^{0mkl} + f_{abc} F^a_{im} F^b_{0n} F^c_{ij} \epsilon^{m0nl} + \\ + f_{abc} F^a_{im} F^b_{n0} F^c_{lj} \epsilon^{mn0l} + f_{abc} F^a_{im} F^b_{nl} F^c_{0j} \epsilon^{mnl0}. \tag{144}$$

Using the fact that the middle two terms are equal, we have

$$g_{ij} = -f_{abc} F^a_{0i} B^l_b \epsilon_{ljm} B^m_c - 2 f_{abc} \epsilon^{mnl} \epsilon_{imk} B^k_a F^b_{0n} \epsilon_{ljp} B^p_c - \\ - 2 f_{abc} \epsilon_{imk} B^k_a B^m_b F^c_{0j}. \tag{145}$$

*We have omitted the conformal factor for simplicity, which can always be re-inserted at the end of the derivations.

Applying epsilon symbol identities to the second term of (145) and simplifying, we get

$$- f_{abc}\,(\det\|B\|)\,F^a_{0i}\,\epsilon_{cbd}\,(B^{-1})^d_j - 2\,f_{abc}\big(\delta^m_j\delta^n_p - \delta^m_p\delta^n_j\big)\times$$
$$\times\,\epsilon_{imk}\,B^k_a F^b_{0n} B^p_c - 2\,f_{abc}\,(\det\|B\|)\,\epsilon_{bad}\,(B^{-1})^d_i F^c_{0j} =$$
$$= 2\,(\det\|B\|)\Big[F^d_{0i}\,(B^{-1})^d_j + F^d_{0j}\,(B^{-1})^d_i\Big] -$$
$$- 2\,(\det\|B\|)\,\epsilon^{nkm}\,(B^{-1})^b_m F^b_{0n}\,\epsilon_{ijk} + 4\,(\det\|B\|)(B^{-1})^d_i F^b_{0j}\,. \quad (146)$$

Note that the third term on the right hand side of (146), upon application of epsilon identities, is given by

$$2\,(\det\|B\|)\big(\delta^n_i\delta^m_j - \delta^n_j\delta^m_i\big)(B^{-1})^b_m F^b_{0n} =$$
$$= 2\,(\det\|B\|)\Big[F^b_{0i}(B^{-1})^b_j - F^b_{0j}(B^{-1})^b_i\Big]. \quad (147)$$

Substituting (147) back into the right had side of (146) and after some cancellations, we get that the spatial part of $g_{\mu\nu}$ is given by

$$g_{ij} = 4\,(\det\|B\|)\Big[F^b_{0i}(B^{-1})^b_j + F^b_{0j}(B^{-1})^b_i\Big], \quad (148)$$

which is symmetric as expected.

Submitted on April 18, 2011

1. Ashtekar A. New Perspectives in Canonical Gravity. Bibliopolis, Napoli, 1988.
2. Ashtekar A. New Hamiltonian formulation of General Relativity. *Phys. Rev. D*, 1987, vol. 36, no. 6, 1587–1602.
3. Ashtekar A. New variables for classical and quantum gravity. *Phys. Rev. Lett.*, 1986, vol. 57, no. 18, 2244–2247.
4. Chakraborty S. and Peldán P. Gravity and Yang-Mills theory: Two faces of the same theory? *Intern. J. Modern Phys. D*, 1994, vol. 3, no. 4, 695–722.
5. Capovilla R., Jacobson T., Dell J. General relativity without the metric. *Class. Quant. Grav.*, 1989, vol. 63, no. 21, 2325–2328.
6. Capovilla R., Dell J., and Jacobson T. A pure spin-connection formulation of gravity. *Class. Quant. Grav.*, 1991, vol. 8, 59–73.
7. Capovilla R., Jacobson T., and Dell J. Gravitational instantons as SU(2) gauge fields. *Class. Quant. Grav.*, 1990, vol. 7, L1–L3.
8. Capovilla R. and Plebanski J. F. Some exact solutions of the Einstein field equations in terms of the self-dual spin connection. *J. Math. Phys.*, 1993, vol. 34, no. 1, 130–138.
9. Lapedes A. and Perry M. Type-D gravitational instantons. *Phys. Rev. D*, 1981, vol. 24, no. 6, 1478–1483.

10. Penrose R. and Rindler W. Spinors and Space-Time. Vol. 1: Two-Spinor Calculus and Relativistic Fields. Cambridge Univ. Press, Cambridge, 1984. Vol. 2: Spinor and Twistor Methods in Space-Time Geometry. Cambridge Univ. Press, Cambridge, 1985.
11. Stephani H., Kramer D., MacCallum M., Hoenselaers C., and Herlt E. Exact Solutions of Einstein's Field Equations. Cambridge Univ. Press, Cambridge, 1980.
12. Plebanski J. F. On the separation of Einsteinian substructures. *J. Math. Phys.*, 1977, vol. 18, no. 12, 2511–2520.
13. Capovilla R., Dell J., and Jacobson T. Self-dual 2-forms and gravity. *Class. Quant. Grav.*, 1991, vol. 8, 41–57.
14. Krasnov K. Plebanski formulation of General Relativity: A practical introduction. Cornell University arXiv: gr-qc/0904.0423.
15. Reisenberger M. New constraints for canonical General Relativity. *Nucl. Phys. B*, 1995, vol. 457, 643 – 687.
16. Urbantke H. On integrability properties of SU(2) Yang-Mills fields. I. Infinitesimal part. *J. Math. Phys.*, 1984, vol. 25, 2321–2324.
17. Peldán P. Actions for gravity, with generalizations: a title. *Class. Quant. Grav.*, 1994, vol. 11, 1087–1132.
18. Samuel J. Gravitational instantons from the Ashtekar variables. *Class. Quant. Grav.*, 1988, vol. 5, L123–L125.
19. Samuel J. A Lagrangian basis for Ashtekar's formulation of canonical gravity. *Pramana J. Phys.*, 1987, vol. 28, no. 4, L429 – L432.
20. Carlip S. Dominant topologies in Euclidean quantum gravity. *Class. Quant. Grav.*, 1998, vol. 15, 2629 – 2638.
21. Kodama H. Holomorphic wavefunction of the Universe. *Phys. Rev. D*, 1990, vol. 42, no. 8, 2548 – 2565.
22. Smolin L. Quantum gravity with a positive cosmological constant. Cornell University arXiv: hep-th/0209079.

Vol. 4, 2011 **ISSN 1654-9163**

THE ABRAHAM ZELMANOV JOURNAL

The journal for General Relativity, gravitation and cosmology

TIDSKRIFTEN ABRAHAM ZELMANOV

Den tidskrift för allmänna relativitetsteorin, gravitation och kosmologi

The Abraham Zelmanov Journal is a non-commercial, academic journal registered with the *Royal National Library of Sweden.* This journal was typeset using LaTeX typesetting system. Powered by BaKoMa-TeX.

The Abraham Zelmanov Journal är en ickekommersiell, akademisk tidskrift registrerat hos *Kungliga biblioteket.* Denna tidskrift är typsatt med typsättningssystemet LaTeX. Utförd genom BaKoMa-TeX.