Cosmological Mass-Defect — A New Effect of General Relativity

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Abstract: This study targets the change of mass of a mass-bearing particle with the distance travelled in the space of the main (principal) cosmological metrics. The mass-defect is obtained due to a new method of deduction: by solving the scalar geodesic equation (equation of energy) of the particle. This equation manifests three factors affecting the particle's mass: gravitation, non-holonomity, and deformation of space. In the space of Schwarzschild's mass-point metric, the obtained solution coincides with the well-known gravitational mass-defect whose magnitude increases toward the gravitating body. No mass-defect has been found in the rotating space of Gödel's metric, and in the space filled with a homogeneous distribution of ideal liquid and physical vacuum (Einstein's metric). The other obtained solutions manifest a mass-defect of another sort than that in the mass-point metric: its magnitude increases with distance from the observer, so that manifests itself at cosmologically large distances travelled by the particle. This effect has been found in the space of Schwarzschild's metric of a sphere of incompressible liquid, in the space of a sphere filled with physical vacuum (de Sitter's metric), and in the deforming spaces of Friedmann's metric (empty or filled with ideal liquid and physical vacuum). Herein, we refer to this effect as the cosmological mass-defect. It has never been considered prior to the present study: it is a new effect of the General Theory of Relativity.

Contents:

§1. Problem statement. In 2008, I presented my theory of the cosmological Hubble redshift [1]. According to the theory, the Hubble redshift was explained as the energy loss of photons with distance due to the work done against the field of global non-holonomity (rotation) of the isotropic space, which is the home of photons*. I arrived at this conclusion after solving the scalar geodesic equation (equation of energy) of a photon travelling in a static (non-deforming) universe. The calculation matched the observed Hubble law, including its non-linearity.

My idea now is that, in analogy to photons, we could as well consider mass-bearing particles.

Let's compare the isotropic and non-isotropic geodesic equations, which are the equations of motion of particles. According to the chronometrically invariant formalism, which was introduced in 1944 by Abraham Zelmanov [3–5], any four-dimensional quantity is observed as its projections onto the time line and three-dimensional spatial section of the observer†. The projected (chronometrically invariant) equations for non-isotropic geodesics have the form [3–5]

$$\frac{dm}{d\tau} - \frac{m}{c^2}\,F_i\,\mathrm{v}^i + \frac{m}{c^2}\,D_{ik}\,\mathrm{v}^i\mathrm{v}^k = 0\,, \tag{1.1}$$

$$\frac{d(m\mathrm{v}^i)}{d\tau} - mF^i + 2m\left(D^i_k + A^{\cdot i}_{k\cdot}\right)\mathrm{v}^k + m\triangle^i_{nk}\,\mathrm{v}^n\mathrm{v}^k = 0\,, \tag{1.2}$$

while the projected equations for isotropic geodesics are

$$\frac{d\omega}{d\tau} - \frac{\omega}{c^2}\,F_i\,c^i + \frac{\omega}{c^2}\,D_{ik}\,c^ic^k = 0\,, \tag{1.3}$$

$$\frac{d(\omega c^i)}{d\tau} - \omega F^i + 2\omega\left(D^i_k + A^{\cdot i}_{k\cdot}\right)c^k + \omega\triangle^i_{nk}\,c^nc^k = 0\,. \tag{1.4}$$

Thus, according to the chronometrically invariant equations of motion, the factors affecting the particles are: the gravitational inertial force F_i, the angular velocity A_{ik} of the rotation of space due to its non-holonomity, the deformation D_{ik} of space, and the non-uniformity of space (expressed by the Christoffel symbols Δ^i_{jk}).

*The four-dimensional pseudo-Riemannian space (space-time) consists of two segregate regions: the non-isotropic space, which is the home of mass-bearing particles, and the isotropic space inhabited by massless light-like particles (photons). The isotropic space rotates with the velocity of light under the conditions of both Special Relativity and General Relativity, due to the sign-alternating property of the space-time metric. See [2] for details.

†Chronometric invariance means that the projected (chronometrically invariant) quantities and equations are invariant along the spatial section of the observer.

As is seen, the non-isotropic geodesic equations have the same form as the isotropic ones. Only the sublight velocity v^i and the relativistic mass m are used instead of the light velocity c^i and the frequency ω of a photon. Therefore, the factors of gravitation, non-holonomity, and deformation, presented in the scalar geodesic equation, should change the mass of a moving mass-bearing particle with distance just as they change the frequency of a photon.

Relativistic mass change due to the field of gravitation of a massive body (the space of Schwarzschild's mass-point metric) is a textbook effect of General Relativity, well verified by experiments. It is regularly deduced from the conservation of energy of a mass-bearing particle in the stationary field of gravitation [6, §88]. However, this method of deduction can only be used in stationary fields [6, §88], wherein gravitation is the sole factor affecting the particle.

In contrast, the new method of deduction of the relativistic mass change with distance I propose herein — through integrating the scalar geodesic equation, based on the chronometically invariant formalism, — is universal. This is because the scalar geodesic equation contains all three factors changing the mass of a moving mass-bearing particle with distance (these are gravitation, non-holonomity, and deformation), and these factors are presented in their general form, without any limitations. Therefore the suggested method of deduction can equally be applied to calculating the relativistic mass change with distance travelled by the particle in any particular space metric known due to the General Theory of Relativity.

In the next paragraphs of this paper, we will apply the suggested method of deduction to the main (principal) cosmological metrics. As a result, we will see how a mass-bearing particle changes its mass with the distance travelled in most of these spaces, including "cosmologically large" distances where the relativistic mass change thus becomes *cosmological mass-defect*.

§2. The chronometrically invariant formalism in brief. Before we solve the geodesic equations in chronometrically invariant form, we need to have a necessary amount of definitions of those quantities specifying the equations. According to the chronometrically invariant formalism [3–5], these are: the chr.inv.-vector of the gravitational inertial force F_i, the chr.inv.-tensor of the angular velocity of the rotation of space A_{ik} due to its non-holonomity (non-orthogonality of the time lines to the three-dimensional spatial section), the chr.inv.-tensor of the deformation of space D_{ik}, and the chr.inv.-Christoffel symbols Δ^i_{jk} (they manifest the non-uniformity of space)

$$F_i = \frac{1}{\sqrt{g_{00}}}\left(\frac{\partial \mathrm{w}}{\partial x^i} - \frac{\partial v_i}{\partial t}\right), \qquad \sqrt{g_{00}} = 1 - \frac{\mathrm{w}}{c^2}\,, \tag{2.1}$$

$$A_{ik} = \frac{1}{2}\left(\frac{\partial v_k}{\partial x^i} - \frac{\partial v_i}{\partial x^k}\right) + \frac{1}{2c^2}\left(F_i v_k - F_k v_i\right), \tag{2.2}$$

$$D_{ik} = \frac{1}{2}\frac{{}^*\partial h_{ik}}{\partial t}\,, \quad D^{ik} = -\frac{1}{2}\frac{{}^*\partial h^{ik}}{\partial t}\,, \quad D = h^{ik}D_{ik} = \frac{{}^*\partial \ln\sqrt{h}}{\partial t}\,, \tag{2.3}$$

$$\Delta^i_{jk} = h^{im}\Delta_{jk,m} = \frac{1}{2}\,h^{im}\left(\frac{{}^*\partial h_{jm}}{\partial x^k} + \frac{{}^*\partial h_{km}}{\partial x^j} - \frac{{}^*\partial h_{jk}}{\partial x^m}\right). \tag{2.4}$$

They are expressed through the chr.inv.-differential operators

$$\frac{{}^*\partial}{\partial t} = \frac{1}{\sqrt{g_{00}}}\frac{\partial}{\partial t}\,, \qquad \frac{{}^*\partial}{\partial x^i} = \frac{\partial}{\partial x^i} + \frac{1}{c^2}\,v_i\,\frac{{}^*\partial}{\partial t}\,, \tag{2.5}$$

as well as the gravitational potential w, the linear velocity v_i of space rotation due to the respective non-holonomity, and also the chr.inv.-metric tensor h_{ik}, which are determined as

$$\mathrm{w} = c^2\left(1 - \sqrt{g_{00}}\right), \qquad v_i = -\frac{c\,g_{0i}}{\sqrt{g_{00}}}\,, \tag{2.6}$$

$$h_{ik} = -\,g_{ik} + \frac{1}{c^2}\,v_i v_k\,, \qquad h^{ik} = -\,g^{ik}, \qquad h^i_k = \delta^i_k\,, \tag{2.7}$$

while the derivation parameter of the equations is the physical observable time

$$d\tau = \sqrt{g_{00}}\,dt - \frac{1}{c^2}\,v_i\,dx^i. \tag{2.8}$$

This is enough. We now have all the necessary equipment to solve the geodesic equations in chronometrically invariant form.

§3. Local mass-defect in the space of a mass-point (Schwarzschild's mass-point metric). This is an empty space*, wherein a spherical massive island of matter is located, thus producing a spherically symmetric field of gravitation (curvature). The massive island is

*In the General Theory of Relativity, we say that a space is empty if it is free of distributed matter — substance or fields, described by the right-hand side of Einstein's equations, — except for the field of gravitation, which is the same as the field of the space curvature described by the left-hand side of the equations.

approximated as a mass-point at distances much larger than its radius. The metric of such a space was introduced in 1916 by Karl Schwarzschild [7]. In the spherical three-dimensional coordinates $x^1 = r$, $x^2 = \varphi$, $x^3 = \theta$, the metric has the form

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right), \qquad (3.1)$$

where r is the distance from the mass-island of the mass M, $r_g = \frac{2GM}{c^2}$ is the corresponding gravitational radius of the mass, and G is the world-constant of gravitation. As is seen from the metric, such a space is free of rotation and deformation. Only the field of gravitation affects mass-bearing particles therein.

Differentiating the gravitational potential $\mathrm{w} = c^2(1 - \sqrt{g_{00}})$ with respect to x^i, we obtain

$$F_i = \frac{1}{\sqrt{g_{00}}} \frac{\partial \mathrm{w}}{\partial x^i} = -\frac{c^2}{2 g_{00}} \frac{\partial g_{00}}{\partial x^i}, \qquad (3.2)$$

wherein, according to the metric (3.1), we should readily substitute

$$g_{00} = 1 - \frac{r_g}{r}. \qquad (3.3)$$

Thus the gravitational inertial force (2.1) in the space of Schwarzschild's mass-point metric has the following nonzero components

$$F_1 = -\frac{c^2 r_g}{2r^2} \frac{1}{1 - \frac{r_g}{r}}, \qquad F^1 = -\frac{c^2 r_g}{2r^2} \qquad (3.4)$$

which, if the mass-island is not a collapsar ($r \gg r_g$), are

$$F_1 = F^1 = -\frac{GM}{r^2}. \qquad (3.5)$$

As a result, the scalar geodesic equation for a mass-bearing particle (1.1) takes the form

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_1 \mathrm{v}^1 = 0, \qquad (3.6)$$

where $\mathrm{v}^1 = \frac{dr}{d\tau}$. This equation transforms into $\frac{dm}{m} = \frac{1}{c^2} F_1 \, dr$, thus we obtain the equation $d \ln m = -\frac{GM}{c^2} \frac{dr}{r^2}$. It solves, obviously, as

$$m = m_0 \, e^{\frac{GM}{c^2 r}} \simeq m_0 \left(1 + \frac{GM}{c^2 r}\right). \qquad (3.7)$$

According to the solution, a spacecraft's mass measured on the surface of the Earth ($M = 6.0 \times 10^{27}$ gram, $r = 6.4 \times 10^{8}$ cm) will be greater than its mass measured at the distance of the Moon ($r = 3.0 \times 10^{10}$ cm) by a value of $1.5 \times 10^{-11}\, m_0$ due to the greater magnitude of the gravitational potential near the Earth.

This mass-defect is a local phenomenon: it decreases with distance from the source of the field, thus becoming negligible at "cosmologically large" distances even in the case of such massive sources of gravitation as the galaxies. This is not a cosmological effect, in other words.

It is known as the gravitational mass-defect in the Schwarzschild mass-point field, which is just one of the basic effects of the General Theory of Relativity. The reason why I speak of this well-known effect herein is that this method of deduction — through integrating the scalar geodesic equation, based on the chronometically invariant formalism, — differs from the regular deduction [6, §88], derived from the conservation of energy of a particle travelling in a stationary field of gravitation.

§4. Local mass-defect in the space of an electrically charged mass-point (Reissner-Nordström's metric). Due to the suggested new method of deduction — through integrating the scalar geodesic equation, based on the chronometically invariant formalism, — we can now calculate mass-defect in the space of Reissner-Nordström's metric. This is a space analogous to the space of the mass-point metric with the only difference being that the spherical massive island of matter is electrically charged: in this case, the massive island is the source of both the gravitational field (the field of the space curvature) and the electromagnetic field. Therefore such a space is not empty but filled with a spherically symmetric electromagnetic field (distributed matter). Such a space has a metric which appears as an actual extension of Schwarzschild's mass-point metric (3.1). The metric was first introduced in 1916 by Hans Reissner [8] then, independently, in 1918 by Gunnar Nordström [9]. It has the form

$$ds^2 = \left(1 - \frac{r_g}{r} + \frac{r_q^2}{r^2}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{r_g}{r} + \frac{r_q^2}{r^2}} - r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \quad (4.1)$$

where r is the distance from the charged mass-island, $r_g = \frac{2GM}{c^2}$ is the corresponding gravitational radius, M is its mass, G is the constant of gravitation, $r_q^2 = \frac{Gq^2}{4\pi\varepsilon_0 c^4}$, where q is the corresponding electric charge, and $\frac{1}{4\pi\varepsilon_0}$ is Coulomb's force constant. As is seen from the metric, such a space is free of rotation and deformation. The gravitational inertial force is, in this case, determined by both Newton's force and Coulomb's force according the component g_{00} of the metric (4.1) which is

$$g_{00} = 1 - \frac{r_g}{r} + \frac{r_q^2}{r^2}\,, \tag{4.2}$$

thus we obtain

$$F_1 = -\frac{c^2}{2\left(1 - \frac{r_g}{r} + \frac{r_q^2}{r^2}\right)}\left(\frac{r_g}{r^2} - \frac{2\,r_q^2}{r^3}\right), \tag{4.3}$$

$$F^1 = -\,-\frac{c^2}{2}\left(\frac{r_g}{r^2} - \frac{2\,r_q^2}{r^3}\right). \tag{4.4}$$

If the massive island is not a collapsar ($r \gg r_g$), and it bears a weak electric charge ($r \gg r_q$), we have

$$F_1 = F^1 = -\frac{c^2}{2}\left(\frac{r_g}{r^2} - \frac{2\,r_q^2}{r^3}\right) = -\frac{GM}{r^2} + \frac{Gq^2}{4\pi\varepsilon_0 c^2}\,\frac{1}{r^3}\,. \tag{4.5}$$

Thus, the scalar geodesic equation for a mass-bearing particle (1.1) takes the form

$$\frac{dm}{d\tau} - \frac{m}{c^2}\,F_1\,\mathrm{v}^1 = 0\,, \tag{4.6}$$

where $\mathrm{v}^1 = \frac{dr}{d\tau}$. It transforms into $d\ln m = \left(-\frac{GM}{c^2 r^2} + \frac{Gq^2}{4\pi\varepsilon_0 c^4}\frac{1}{r^3}\right)dr$, which solves, obviously, as

$$m = m_0\, e^{\frac{GM}{c^2 r} - \frac{1}{2r^2}\frac{Gq^2}{4\pi\varepsilon_0 c^4}} \simeq m_0\left(1 + \frac{GM}{c^2 r} - \frac{1}{2r^2}\,\frac{Gq^2}{4\pi\varepsilon_0 c^4}\right). \tag{4.7}$$

As is seen from the solution, we should expect a mass-defect to be observed in the space of Reissner-Nordström's metric. Its magnitude is that of the mass-defect of the mass-point metric (the second term in the solution) with a second-order correction — the mass-defect due to the electromagnetic field of the massive island (the third term). The magnitude of the correction decreases with distance from the source of the field (a charged spherical massive island) even faster than the mass-defect due to the field of gravitation of the massive island. Therefore, the mass-defect in the space of Reissner-Nordström's metric we have obtained here is a local phenomenon, not a cosmological effect.

Note that this is the first case, where a mass-defect is predicted due to the presence of the electromagnetic field. Such an effect was not considered in the General Theory of Relativity prior to the present study.

A note concerning two other primary extensions of Schwarzschild's mass-point metric. Kerr's metric describes the space of a rotating mass-point. It was introduced in 1963 by Roy P. Kerr [10] then transformed into suitable coordinates by Robert H. Boyer and Richard W. Lindquist [11]. The Kerr-Newman metric was introduced in 1965 by Ezra T. Newman [12,13]. It describes the space of a rotating, electrically charged mass-point. These metrics are deduced in the vicinity of the point-like source of the field: they do not contain the distribution function of the rotational velocity with distance from the source. As a result, when taking into account the geodesic equations to be integrated in the space of any one of the rotating mass-point metrics, we should introduce the functions on our own behalf. This is not good at all: our choice of the functions, based on our understanding of the space rotation, can be true or false. We therefore omit calculation of mass-defect in the space of a rotating mass-point (Kerr's metric), and in the space of a rotating, electrically charged mass-point (the Kerr-Newman metric).

§5. No mass-defect present in the rotating space with self-closed time-like geodesics (Gödel's metric). This space metric was introduced in 1949 by Kurt Gödel [14], in order to find a possibility of time travel (realized through self-closed time-like geodesics). Gödel's metric, as was shown by himself [14], satisfies Einstein's equations where the right-hand side contains the energy-momentum tensor of dust and also the λ-term. This means that such a space is not empty, but filled with dust and physical vacuum (λ-field). Also, it rotates so that time-like geodesics are self-closed therein. Gödel's metric in its original notation, given in his primary publication [14], is

$$ds^2 = a^2\left[(d\tilde{x}^0)^2 + 2\,e^{\tilde{x}^1} d\tilde{x}^0 d\tilde{x}^2 - (d\tilde{x}^1)^2 + \frac{e^{2\tilde{x}^1}}{2}\,(d\tilde{x}^2)^2 - (d\tilde{x}^3)^2\right], \quad (5.1)$$

where $a = const > 0$ [cm] is a constant of the space, determined through Einstein's equations as $\lambda = -\frac{1}{2a^2} = -\frac{4\pi G\rho}{c^2}$ so that $a^2 = \frac{c^2}{8\pi G\rho} = -\frac{1}{2\lambda}$, and ρ is the dust density. Gödel's metric in its original notation (5.1) is expressed through the dimensionless Cartesian coordinates $d\tilde{x}^0 = \frac{1}{a}\,dx^0$, $d\tilde{x}^1 = \frac{1}{a}\,dx^1$, $d\tilde{x}^2 = \frac{1}{a}\,dx^2$, $d\tilde{x}^3 = \frac{1}{a}\,dx^3$, which emphasize the meaning of the world-constant a of such a space. Also, this is a constant-curvature space wherein the curvature radius is $R = \frac{1}{a^2} = const > 0$.

We now move to the regular Cartesian coordinates $a d\tilde{x}^0 = dx^0 = c\,dt$, $a d\tilde{x}^1 = dx^1$, $a d\tilde{x}^2 = dx^2$, $a d\tilde{x}^3 = dx^3$, which are more suitable for the calculation of the components of the fundamental metric tensor, thus manifesting the forces acting in the space better. As a result, we obtain Gödel's metric in the form

$$ds^2 = c^2dt^2 + 2e^{\frac{x^1}{a}}cdt\,dx^2 - (dx^1)^2 + \frac{e^{\frac{2x^1}{a}}}{2}(dx^2)^2 - (dx^3)^2. \qquad (5.2)$$

As is seen from this form of Gödel's metric,

$$g_{00} = 1\,, \qquad g_{02} = e^{\frac{x^1}{a}}, \qquad g_{01} = g_{03} = 0\,, \qquad (5.3)$$

thus implying that such a space is free of gravitation, but rotates with a three-dimensional linear velocity v_i (determined by g_{0i}) whose only nonzero component is v_2. The velocity v_2 (actually, the component g_{02}) manifests the cosine of the angle of inclination of the line of time $x^0 = ct$ to the spatial axis $x^2 = y$. Therefore the lines of time are non-orthogonal to the spatial axis at each single point of a Gödel space, owing to which local time-like geodesics are the elements of big circles (self-closing time-like geodesics) therein. The nonzero v_2 also means that the shift of the whole three-dimensional space along the axis draws a big circle. This velocity, according to the definition of v_i (2.6) provided by the chronometrically invariant formalism, is

$$v_2 = -\,ce^{\frac{x^1}{a}}, \qquad (5.4)$$

which, obviously, does not depend on time. Therefore, in the space of Gödel's metric, the second (inertial) term of the gravitational inertial force F_i (2.1) is zero as well as the first (gravitational) term. The metric is also free of deformation: the spatial components g_{ik} of the fundamental metric tensor do not depend on time therein.

As a result, we see that no one of the factors changing the mass of a mass-bearing particle according to the scalar geodesic equation (whose factors are gravitation, non-holonomity, and deformation of space) is present in the space of Gödel's metric. We therefore conclude that mass-bearing particles do not achieve mass-defect with the distance travelled in a Gödel universe.

§6. Cosmological mass-defect in the space of Schwarzschild's metric of a sphere of incompressible liquid. This is the internal space of a sphere filled, homogeneously, with an incompressible liquid. The preliminary form metric of such a space was introduced in 1916 by Karl Schwarzschild [15]. He however limited himself to the assumption that the three-dimensional components of the fundamental metric tensor should not possess breaking (discontinuity). The general form of this metric, which is free of this geometric limitation, was deduced in 2009 by Larissa Borissova: see formula (3.55) in [16], or (1.1) in [17]. It is

$$ds^2 = \frac{1}{4}\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)^2 c^2dt^2 - \\ - \frac{dr^2}{1-\frac{\varkappa\rho_0 r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \qquad (6.1)$$

where $\varkappa = \frac{8\pi G}{c^2}$ is Einstein's gravitational constant, $\rho_0 = \frac{M}{V} = \frac{3M}{4\pi a^3}$ is the density of the liquid, a is the sphere's radius, and r is the radial coordinate from the central point of the sphere. The metric manifests that such a space is free of rotation and deformation. Only gravitation affects mass-bearing particles therein. It is determined by

$$g_{00} = \frac{1}{4}\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)^2. \qquad (6.2)$$

Respectively, the gravitational inertial force (2.1) in the space of the generalized Schwarzschild metric of a sphere of incompressible liquid has the following nonzero components

$$F_1 = -\frac{c^2\varkappa\rho_0 r}{3\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)}, \qquad (6.3)$$

$$F^1 = -\frac{c^2\varkappa\rho_0 r\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}{3\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)}, \qquad (6.4)$$

while the remaining components of the force are zero, because, as is seen from the metric (6.1), the component g_{00}, which determines the force, is only dependent on the radial coordinate $x^1 = r$.

Thus the scalar geodesic equation for a mass-bearing particle (1.1) takes the form

$$\frac{dm}{d\tau} - \frac{m}{c^2}F_1\,\mathrm{v}^1 = 0\,, \qquad (6.5)$$

where $\mathrm{v}^1 = \frac{dr}{d\tau}$, while F_1 is determined by (6.3). This equation transforms, obviously, into $d\ln m = \frac{1}{c^2}F_1\,dr$, thus

$$d\ln m = -\frac{\varkappa\rho_0 r}{3\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,\frac{dr}{\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}} - \sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)}. \qquad (6.6)$$

Meanwhile,

$$d\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}-\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right)=\frac{\varkappa\rho_0 r}{3}\frac{dr}{\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,, \tag{6.7}$$

therefore the initial equation transforms into

$$d\ln m=-\,d\ln\left(3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}-\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}\right), \tag{6.8}$$

which solves as

$$m=m_0\,\frac{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}-1}{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}-\sqrt{1-\frac{\varkappa\rho_0 r^2}{3}}}\,. \tag{6.9}$$

Because the world-density is quite small, $\rho_0 \approx 10^{-29}$ gram/cm^3 or even less than it, and Einstein's gravitational constant is very small as well, $\varkappa = \frac{8\pi G}{c^2} = 1.862\times 10^{-27}$ cm/gram, the obtained solution (6.9) at distances much smaller than the radius of such a universe ($r \ll a$), takes the simplified form

$$m=m_0\left(1-\frac{\varkappa\rho_0 r^2}{12}\right). \tag{6.10}$$

As such, mass-defect in a spherical universe filled with incompressible liquid is negative. The magnitude of the negative mass-defect increases with distance from the observer, eventually taking the ultimately high numerical value at the event horizon. Hence, this is definitely a true instance of cosmological effects. We will therefore further refer to this effect as the *cosmological mass-defect*.

In other words, the more distant an object we observe in such a universe is, the less is its observed mass in comparison to its real rest-mass measured near this object.

If our Universe would be a sphere of incompressible liquid, the mass-defect would be negligible within our Galaxy "Milky Way" (because ρ_0 and $\varkappa$ are very small). However, it would become essential at distances of even the closest galaxies: an object located as distant as the Andromeda Galaxy ($r \simeq 780\times 10^3$ pc $\simeq 2.4\times 10^{24}$ cm) would have a negative cosmological mass-defect equal, according to the linearized solution (6.10), to $\frac{\varkappa\rho_0 r^2}{12} \approx 10^{-8}$ of its true rest-mass m_0.

At the ultimate large distance in such a universe, which is the event horizon $r = a$, the obtained solution (6.9) manifests the ultimately high mass-defect

$$m = m_0 \, \frac{3\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}-1}{2\sqrt{1-\frac{\varkappa\rho_0 a^2}{3}}} \,. \tag{6.11}$$

§7. Cosmological mass-defect in the space of a sphere filled with physical vacuum (de Sitter's metric). Such a space was first considered in 1917 by Willem de Sitter [18,19]. It contains no substance, but is filled with a spherically symmetric distribution of physical vacuum (λ-field). Its curvature is constant at each point: this is a constant-curvature space. Its metric, introduced by de Sitter, is

$$ds^2 = \left(1-\frac{\lambda r^2}{3}\right) c^2 dt^2 - \frac{dr^2}{1-\frac{\lambda r^2}{3}} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{7.1}$$

which contains the λ-term of Einstein's equations. Such a space is as well free of rotation and deformation, while gravitation is only determined by the λ-term

$$g_{00} = 1 - \frac{\lambda r^2}{3} \,. \tag{7.2}$$

Respectively, the sole nonzero components of the gravitational inertial force (2.1) in such a space are

$$F_1 = \frac{\lambda c^2}{3}\frac{r}{1-\frac{\lambda r^2}{3}} \,, \qquad F^1 = \frac{\lambda c^2}{3}\, r \,, \tag{7.3}$$

while the remaining ones are zero: the component g_{00}, which determines gravitation, in de Sitter's metric (7.1) is dependent only on the radial coordinate $x^1 = r$. This is a non-Newtonian gravitational force which is due to the λ-field (physical vacuum). Its magnitude increases with distance: if $\lambda < 0$, this is a force of attraction, if $\lambda > 0$ this is a force of repulsion.

Thus the scalar geodesic equation for a mass-bearing particle (1.1) in this case has the form

$$\frac{dm}{d\tau} - \frac{m}{c^2} F_1 \mathrm{v}^1 = 0 \,, \tag{7.4}$$

where $\mathrm{v}^1 = \frac{dr}{d\tau}$, with F_i determined by (7.3). It transforms, obviously, into $d\ln m = \frac{1}{c^2} F_1\, dr$, which is

$$d\ln m = \frac{\lambda r}{3}\frac{dr}{1-\frac{\lambda r^2}{3}} \,. \tag{7.5}$$

Because

$$d \ln \left(1 - \frac{\lambda r^2}{3}\right) = -\frac{2\lambda r}{3} \frac{dr}{1 - \frac{\lambda r^2}{3}}\,, \tag{7.6}$$

the initial equation takes the form

$$d \ln m = -\frac{1}{2}\, d \ln \left(1 - \frac{\lambda r^2}{3}\right), \tag{7.7}$$

which solves as

$$m = \frac{m_0}{\sqrt{1 - \frac{\lambda r^2}{3}}}\,. \tag{7.8}$$

Because, according to astronomical estimates, the λ-term is quite small as $\lambda \leqslant 10^{-56}\,\text{cm}^{-2}$, at small distances this solution becomes

$$m = m_0 \left(1 + \frac{\lambda r^2}{6}\right). \tag{7.9}$$

As is seen from the obtained solution, a positive mass-defect should be observed in a de Sitter universe: the more distant the observed object therein is, the greater is its observed mass in comparison to its real rest-mass measured near the object. The magnitude of this effect increases with distance with respect to the object under observation. In other words, this is another *cosmological mass-defect*.

For instance, suppose our Universe to be a de Sitter world. Consider an object, which is located at the distance of the Andromeda Galaxy ($r \simeq 780 \times 10^3\,\text{pc} \simeq 2.4 \times 10^{24}\,\text{cm}$). In this case, with $\lambda \leqslant 10^{-56}\,\text{cm}^{-2}$ and according to the linearized solution (7.9), the mass of this object registered in our observation should be greater than its true rest-mass m_0 for a value of $\frac{\lambda r^2}{6} \leqslant 10^{-8}$. However, at the event horizon $r \approx 10^{28}\,\text{cm}$, which is the ultimately large distance observed in our Universe according to the newest data of observational astronomy, the magnitude of the mass-defect, according to the obtained exact solution (7.8), is expected to be very high, even approaching infinity.

Therefore, the one of experimenta crucis answering the question "is our Universe a de Sitter world or not?" would be a substantially high positive mass-defect of distant galaxies and quasars.

§8. No mass-defect present in the space of a sphere filled with ideal liquid and physical vacuum (Einstein's metric). This cosmological solution was introduced by Albert Einstein in his famous presentation [20], held on February 8, 1917, wherein he introduced relativistic cosmology. This solution implies a closed spherical space, which is filled with homogeneous and isotropic distribution of ideal (non-viscous) liquid and physical vacuum (λ-field). It was not the first of the exact solutions of Einstein's equations, found by the relativists, but the *first cosmological model* — this metric was suggested (by Einstein) as the most suitable model of the Universe as a whole, answering the data of observational astronomy known in those years. The metric of such a space, known also as Einstein's metric, has the form

$$ds^2 = c^2 dt^2 - \frac{dr^2}{1-\lambda r^2} - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{8.1}$$

which is similar to de Sitter's metric (7.1), with only the difference being that Einstein's metric has $g_{00}=1$ and there is no numerical coefficient $\frac{1}{3}$ in the denominator of g_{11}. Herein $\lambda = \frac{4\pi G\rho}{c^2}$, i.e. the cosmological λ-term has the opposite sign compared to that of Gödel's metric.

As is seen, in Einstein's metric,

$$g_{00} = 1\,, \qquad g_{01} = g_{02} = g_{03} = 0\,, \tag{8.2}$$

thus implying that such a space is free of gravitation and rotation. It is also not deforming: the three-dimensional components g_{ik} do not depend on time therein. So, the metric contains no one of the factors changing the mass of a mass-bearing particle according to the scalar geodesic equation. This means that mass-bearing particles do not achieve mass-defect with the distance travelled in the space of Einstein's metric.

§9. Cosmological mass-defect in the deforming spaces of Friedmann's metric. This space metric was introduced in 1922 by Alexander Friedmann as a class of non-stationary solutions to Einstein's equations aimed at generalizing the static homogeneous, and isotropic cosmological model suggested in 1917 by Einstein. Spaces of Friedmann's metric can be empty, or filled with a homogeneous and isotropic distribution of ideal (non-viscous) liquid in common with physical vacuum (λ-field), or filled with one of the media. In a particular case, it can be dust. This is because the energy-momentum tensor of ideal liquid transforms into the energy-momentum tensor of dust by removing the term containing pressure (in this sense, dust behaves as pressureless ideal liquid).

Friedmann's metric in the spherical three-dimensional coordinates has the form

$$ds^2 = c^2 dt^2 - R^2\left[\frac{dr^2}{1-\kappa r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right], \tag{9.1}$$

where $R = R(t)$ is the curvature radius of the space, while $\kappa = 0, \pm 1$ is the curvature factor*. In the case of $\kappa = -1$, the four-dimensional space curvature is negative: this manifests an open three-dimensional space of the hyperbolic type. The case of $\kappa = 0$ yields zero curvature (flat three-dimensional space). If $\kappa = +1$, the four-dimensional curvature is positive, giving a closed three-dimensional space of the elliptic type.

The non-static cosmological models with $\kappa = +1$ and $\kappa = -1$ were considered in 1922 by Friedmann in his primary publication [21] wherein he pioneered non-stationary solutions of Einstein's equations, then in 1924, in his second (last) paper [22]. However, the most popular among the cosmologists is the generalized formulation of Friedmann's metric, which contains all three cases $\kappa = 0, \pm 1$ of the space curvature as in (9.1). It was first considered in 1925 by Georges Lemaître [23, 24], who did not specify κ, then in 1929 by Howard Percy Robertson [25], and in 1937 by Arthur Geoffrey Walker [26]. Friedmann's metric in its generalized form (9.1) containing $\kappa = 0, \pm 1$ is also conventionally known as the Friedmann-Lemaître-Robertson-Walker metric.

A short note about the dimensionless radial coordinate r used in Friedmann's metric (9.1). In a deforming (expanding or compressing) space, the regular coordinates change their scales with time. In particular, if the space deforms as any expanding or compressing spherical space, the regular radial coordinate will change its scale. To remove this problem, Friedmann's metric is regularly expressed through a "homogeneous" radial coordinate r as in (9.1)†. It comes as the regular radial coordinate (circumference measured on the hypersphere), which is then divided by the curvature radius whose scale changes with time accordingly. As a result, the homogeneous radial coordinate r ("reduced" circumference) does not change its scale with time during expansion or compression of the space.

Let's have a look at Friedmann's metric (9.1). We see that

$$g_{00} = 1\,, \qquad g_{0i} = 0\,, \qquad g_{ik} = g_{ik}(t)\,, \tag{9.2}$$

hence, such a space is free of gravitation and rotation, while its three-dimensional subspace deforms. Therefore, the scalar geodesic equation

*This form of Friedmann's metric, containing the curvature factor κ, was introduced due to the independent studies conducted by Lemaître [23, 24] and Robertson [25], following Friedmann's death in 1925.

†Sometimes, Cartesian coordinates are more reasonable for the purpose of calculation. In this case, Friedmann's metric is expressed through the "homogeneous" Cartesian coordinates, which are derived in the same way from the regular Cartesian coordinates, and which are also dimensionless. See Zelmanov's book on cosmology [4] and his paper [5], for instance.

(1.1) for a mass-bearing particle which travels in the space of Friedmann's metric (we assume that it travels along the radial coordinate r with respect to the observer) takes the form

$$\frac{dm}{d\tau} + \frac{m}{c^2}\, D_{11}\, \mathrm{v}^1 \mathrm{v}^1 = 0\,, \tag{9.3}$$

where $\mathrm{v}^1 = \frac{dr}{d\tau}$ [sec^{-1}], while only the space deformation along the radial coordinate, which is D_{11}, affects the mass of the particle during its motion. According to Friedmann's metric, $d\tau = dt$ due to $g_{00} = 1$ and $g_{0i} = 0$. Thus the scalar geodesic equation (9.3) transforms into

$$d\ln m = -\frac{1}{c^2}\, D_{11}\, \dot{r}^2\, dt\,. \tag{9.4}$$

Unfortunately, this equation, (9.4), cannot be solved alone, as well as the scalar geodesic equation in any deforming space: the deformation term of the equation contains the velocity of the particle which is unknown and is determined by the space metric. We find the velocity from the vectorial geodesic equation (1.2), which for a mass-bearing particle travelling in the radial direction r in the space of Friedmann's metric (9.1) takes the form

$$\frac{d\mathrm{v}^1}{d\tau} + \frac{1}{m}\frac{dm}{d\tau}\, \mathrm{v}^1 + 2\, D_1^1\, \mathrm{v}^1 + \triangle_{11}^1\, \mathrm{v}^1 \mathrm{v}^1 = 0\,. \tag{9.5}$$

To remove m from the vectorial geodesic equation (9.5), we make a substitution of the scalar equation (9.3). We obtain a second-order differential equation with respect to r, which has the form

$$\ddot{r} + 2\, D_1^1\, \dot{r} + \Delta_{11}^1\, \dot{r}^2 - \frac{1}{c^2}\, D_{11}\, \dot{r}^3 = 0\,. \tag{9.6}$$

According to the definitions of D_{ik} (2.3) and Δ_{ik}^i (2.4), we calculate D_{11}, D_1^1, and Δ_{11}^1 in the space of Friedmann's metric. To do it, we use the components of the chr.inv.-metric tensor h_{ik} (2.7) calculated according to Friedmann's metric (9.1). After some algebra, we obtain

$$h_{11} = \frac{R^2}{1 - \kappa r^2}\,, \quad h_{22} = R^2 r^2, \quad h_{33} = R^2 r^2 \sin^2\theta\,, \tag{9.7}$$

$$h = \det\|h_{ik}\| = h_{11}\, h_{22}\, h_{33} = \frac{R^6 r^4 \sin^2\theta}{1 - \kappa r^2}\,, \tag{9.8}$$

$$h^{11} = \frac{1 - \kappa r^2}{R^2}\,, \quad h^{22} = \frac{1}{R^2 r^2}\,, \quad h^{33} = \frac{1}{R^2 r^2 \sin^2\theta}\,. \tag{9.9}$$

As a result, we obtain, in the general case of an arbitrary space of Friedmann's metric,

$$D_{11} = \frac{R}{1-\kappa r^2}\frac{\partial R}{\partial t} = \frac{R\dot{R}}{1-\kappa r^2}\,, \quad D_1^1 = \frac{\dot{R}}{R}\,, \quad D = \frac{3\dot{R}}{R}\,, \tag{9.10}$$

$$\Delta_{11}^1 = \frac{\kappa r}{1-\kappa r^2}\,, \tag{9.11}$$

thus our equation (9.6) takes the form

$$\ddot{r} + \frac{2\dot{R}}{R}\,\dot{r} + \frac{\kappa r}{1-\kappa r^2}\,\dot{r}^2 - \frac{R\dot{R}}{c^2\left(1-\kappa r^2\right)}\,\dot{r}^3 = 0\,. \tag{9.12}$$

This equation is non-solvable being considered in the general form as here. To solve this equation, we should simplify it by assuming particular forms of the functions κ and $R = R(t)$.

The curvature factor κ can be chosen very easily: with $\kappa = 0$ we have a deforming flat universe, $\kappa = +1$ describes a deforming closed universe, while $\kappa = -1$ means a deforming open universe.

The curvature radius as a function of time, $R = R(t)$, appears due to that fact that the space deforms. This function can be found through the tensor of the space deformation D_{ik}, whose trace

$$D = h^{ik} D_{ik} = \frac{{}^*\partial \ln\sqrt{h}}{\partial t} = \frac{1}{\sqrt{h}}\frac{{}^*\partial\sqrt{h}}{\partial t} = \frac{1}{V}\frac{{}^*\partial V}{\partial t} \tag{9.13}$$

yields the speed of relative deformation (expansion or compression) of the volume of the space element [4, 5]. The volume of a space element, which plays the key rôle in the formula, is calculated as follows. A parallelepiped built on the vectors $r^i_{(1)}, r^i_{(2)}, \ldots, r^i_{(n)}$ in an n-dimensional Euclidean space has its volume calculated as $V = \pm \det \| r^i_{(n)} \| = \pm | r^i_{(n)} |$. We thus have an invariant $V^2 = | r^i_{(n)} || r_{(m)\,i} | = | r^i_{(n)} || h_{ik}\, r^k_{(m)} | = | h_{ik}\, r^i_{(n)} r^k_{(m)} |$, where $h_{ik} \equiv -g_{ik}$ according to Euclidean geometry. Thus, we obtain $(dV)^2 = | h_{ik}\, dx^i_{(n)} dx^k_{(m)} | = | h_{ik} || dx^i_{(n)} || dx^k_{(m)} | = h\, | dx^i_{(n)} || dx^k_{(m)} |$. Finally, we see that the volume of a differentially small element of an Euclidean space is calculated as $dV = \sqrt{h}\, | dx^i_{(n)} |$. Extending this method into a Riemannian space such as the basic space (space-time) of the General Theory of Relativity, we obtain $dV = \sqrt{-g}\, | dx^\alpha_{(\nu)} |$. In particular, the volume of a three-dimensional (spatial) differentially small element therein is $dV = \sqrt{h}\, | dx^i_{(n)} |$, or, if the parallelepiped's edges meet the (curved) spatial coordinate axes, $dV = \sqrt{h}\, dx^1 dx^2 dx^3$. The total volume of an extended space element is a result of integration of dV along all three spatial coordinates. Thus, in an arbitrary three-dimensional space, which is a subspace of the entire space-time, we obtain

$$D = \frac{{}^*\partial \ln\sqrt{h}}{\partial t} = \frac{1}{\sqrt{h}}\frac{{}^*\partial\sqrt{h}}{\partial t} = \frac{1}{V}\frac{{}^*\partial V}{\partial t} = \gamma\,\frac{1}{a}\frac{{}^*\partial a}{\partial t} = \gamma\,\frac{u}{a}\,, \tag{9.14}$$

where a is the radius of the extended volume ($V \sim a^3$), u is the linear velocity of its deformation (positive in the case of expansion, and negative in the case of compression), and $\gamma = const$ is the shape factor of the space ($\gamma = 3$ in the homogeneous isotropic models [4, 5]).

Taking this formula into account, I would like to introduce two main types of the corresponding space deformation, and two respective types of the function $R = R(t)$. They are as follows.

A constant-deformation (homotachydioncotic) universe. Each single volume V of such a universe, including its total volume and differential volumes, undergoes equal relative changes with time*

$$D = \frac{1}{V}\frac{{}^*\partial V}{\partial t} = \gamma\,\frac{u}{a} = const\,. \tag{9.15}$$

If such a universe expands, the linear velocity of the expansion increases with time. This is an accelerated expanding universe. In contrast, if such a universe compresses, the linear velocity of its compression decreases with time: this is a decelerated compressing universe.

In spaces of Friedmann's metric, $D = \frac{3\dot{R}}{R}$ (9.10). Once $\frac{\dot{R}}{R} = A = const$ that means $D = const$, we have $\frac{1}{R}\,dR = A dt$ that means $d\ln R = A dt$. As a result, denoting $R_0 = a_0$, we obtain that

$$R = a_0\, e^{At}, \qquad \dot{R} = a_0\, A\, e^{At} \tag{9.16}$$

in this case. Substituting the solutions into the general formulae (9.10), we obtain that, in a constant deformation Friedmann universe,

$$D = \frac{3\dot{R}}{R} = 3A = const\,, \tag{9.17}$$

$$D_{11} = \frac{R\dot{R}}{1-\kappa r^2} = \frac{a_0^2\, A\, e^{2At}}{1-\kappa r^2}\,, \tag{9.18}$$

$$D_1^1 = \frac{\dot{R}}{R} = A = const\,. \tag{9.19}$$

*I refer to this kind of universe as *homotachydioncotic* (ομοταχυδιογκωτικό). This terms originates from *homotachydioncosis* — ομοταχυδιόγκωσης — volume expansion with a constant speed, from όμο which is the first part of όμοιος (omeos) — the same, ταχύτητα — speed, διόγκωση — volume expansion, while compression can be considered as negative expansion.

A constant speed deforming (homotachydiastolic) universe. Such a universe deforms with a constant linear velocity* $u = \frac{{}^*\partial a}{\partial t} = const.$ As a result, the radius of any volume element changes linearly with time $a = a_0 + ut$ (the sign of u is positive in an expanding universe, and negative in the case of compression). Thus, relative change of such a volume is expressed, according to the general formula (9.14), as

$$D = \gamma \frac{u}{a_0 + ut} \simeq \gamma \frac{u}{a_0}\left(1 - \frac{ut}{a_0}\right). \tag{9.20}$$

We see that deformation of such a universe decreases with time in the case of expansion, and increases with time if it compresses.

With $D = \frac{\gamma u}{a_0 + ut}$ (9.20), because $D = \frac{3\dot{R}}{R}$ in spaces of Friedmann's metric, we arrive at the simplest equation $\frac{3}{R}\,dR = \frac{\gamma u}{a_0 + ut}\,dt$. It obviously solves, in the Friedmann case ($\gamma = 3$), as $R = a_0 + ut$. Thus we obtain

$$R = a_0 + ut\,, \qquad \dot{R} = u\,. \tag{9.21}$$

As a result, substituting the solutions into the general formulae (9.10), we obtain, in a constant-speed deforming Friedmann universe,

$$D = \frac{3\dot{R}}{R} = \frac{3u}{a_0 + ut}\,, \tag{9.22}$$

$$D_{11} = \frac{R\dot{R}}{1 - \kappa r^2} = \frac{(a_0 + ut)\,u}{1 - \kappa r^2}\,, \tag{9.23}$$

$$D_1^1 = \frac{\dot{R}}{R} = \frac{u}{a_0 + ut}\,. \tag{9.24}$$

In reality, space expands or compresses as a whole so that its volume undergoes equal relative changes with time. Therefore, if our Universe really deforms — expands or compresses — it is a space of the homotachydioncotic (constant deformation) kind. Therefore, we will further consider a constant-deformation Friedmann universe as follows.

Consider the vectorial geodesic equation (9.12) in the simplest case of Friedmann universe, wherein $\kappa = 0$. This is a flat three-dimensional space which expands or compresses due to the four-dimensional curvature which, having a radius R, is nonzero. In such a Friedmann universe

*I refer to this kind of universe as *homotachydiastolic* (ομοταχυδιαστολικός). Its origin is *homotachydiastoli* — ομοταχυδιαστολή — linear expansion with a constant speed, from όμο which is the first part of όμοιος — the same, ταχύτητα — speed, and διαστολή — linear expansion (compression is the same as negative expansion).

($\kappa = 0$, $D = 3A = const$), while taking into account that under the condition of constant deformation we have $R = a_0 e^{At}$ and $\dot{R} = a_0 A e^{At}$ (9.16), the vectorial geodesic equation (9.12) takes the most simplified form

$$\ddot{r} - \frac{a_0^2 A e^{2At}}{c^2}\,\dot{r}^3 + 2A\dot{r} = 0\,. \tag{9.25}$$

Let's introduce a new variable $\dot{r} \equiv p$. Thus $\ddot{r} = \frac{dr}{dt}\frac{dp}{dr} = pp'$, where $p' = \frac{dp}{dr}$. Thus re-write the initially equation (9.25) with the new variable. We obtain

$$pp' - \frac{a_0^2 A e^{2At}}{c^2}\,p^3 + 2Ap = 0\,. \tag{9.26}$$

Assuming that $p \neq 0$, we reduce this equation by p. We obtain

$$p' - \frac{a_0^2 A e^{2At}}{c^2}\,p^2 + 2A = 0\,. \tag{9.27}$$

By introducing the denotations $a = -\frac{a_0^2 A e^{2At}}{c^2}$ and $b = -2A$ we transform this equation into the form

$$p' + ap^2 = b\,. \tag{9.28}$$

This is Riccati's equation: see Kamke [27], Part III, Chapter I, §1.23. We assume a natural condition that $ab = \frac{2a_0^2 A^2 e^{2At}}{c^2} > 0$. The solution of Riccati's equation under $ab > 0$, and with the initially conditions $\xi \equiv r(t_0)$ and $\eta \equiv \dot{r}_0 = \dot{r}(t_0)$, is

$$\dot{r} = p = \frac{\dot{r}_0\sqrt{ab} + b\tanh\sqrt{ab}\,(r - r_0)}{\sqrt{ab} + a\dot{r}_0\tanh\sqrt{ab}\,(r - r_0)}\,, \tag{9.29}$$

where we immediately assume $r(t_0) = 0$ and $\dot{r}_0 = \dot{r}(t_0) = 0$, then extend the variables a and b according to our denotations. We obtain

$$\dot{r} = \frac{br\tanh\sqrt{ab}}{\sqrt{ab}} = \frac{\sqrt{2}\,cr}{a_0 e^{At}}\tanh\frac{\sqrt{2}\,a_0 A e^{At}}{c}\,. \tag{9.30}$$

Let's now substitute this solution into the initial scalar geodesic equation (9.4). We obtain

$$d\ln m = -2Ar^2\tanh^2\left(\frac{\sqrt{2}\,a_0 A e^{At}}{c}\right)dt\,, \tag{9.31}$$

thus we arrive at an integral which has the form

$$\ln m = -2A\int r^2\tanh^2\left(\frac{\sqrt{2}\,a_0 A e^{At}}{c}\right)dt + B\,, \quad B = const. \tag{9.32}$$

This integral in non-solvable. We can only qualitatively study it. So... the solution should have the following form:

$$m = m_0\, e^{-2A\int r^2 \tanh^2\left(\frac{\sqrt{2}\, a_0 A\, e^{At}}{c}\right) dt}. \tag{9.33}$$

We see that, in an expanding Friedmann universe ($A > 0$), the particle's mass m decreases, exponentially, with the distance travelled by it. In a compressing Friedmann universe ($A < 0$), the mass increases, exponentially, according to the travelled distance. In any case, the magnitude of the mass-defect increases with distance from the object under observation. So, this is another instance of *cosmological mass-effect*.

So, we have obtained that cosmological mass-defect should clearly manifest in the space of even the simplest Friedmann metric. Experimental verification of this theoretical conclusion should manifest whether, after all, we live in a Friedmann universe or not.

The vectorial geodesic equation (9.12) with $\kappa = +1$ or $\kappa = -1$ is much more complicated than the most simplified equation (9.25) we have considered in the case of $\kappa = 0$. It leads to integrals which are not only non-solvable by exact methods, but also hard-to-analyze in the general form (without simplification). Therefore, I see two practical ways of considering cosmological mass-defect in the closed and open Friedmann universes ($\kappa = \pm 1$, respectively). First, the consideration of a very particular case of such a universe, with many simplifications and artificially determined functions. Second, the application of computer-aided numerical methods. Anyhow, these allusions are beyond the scope of this principal study.

§10. Conclusions. As is well-known, mass-defect due to the field of gravitation is regularly attributed to the generally covariant formalism, which gives a deduction of it through the conservation of the energy of a particle moving in a stationary field of gravitation [6, §88]. In other words, this well-known effect is regularly considered per se.

In contrast, the chronometrically invariant formalism manifests the gravitational mass-defect as one instance in the row of similar effects, which can be deduced as a result of integrating the scalar geodesic equation (equation of energy) of a mass-bearing particle. This new method of deduction has been suggested herein. It is not limited to the very particular case of the Schwarzschild mass-point field as is the case of the aforementioned old method. The new method can be applied to a particle travelling in the space of any metric theoretically conceivable due to the General Theory of Relativity.

Herein, we have successfully applied this new method of deduction to the main (principal) cosmological metrics.

In the space of Schwarzschild's mass-point metric, the obtained solution coincides with the known gravitational mass-defect [6, §88] whose magnitude increases toward the gravitating body. A similar effect has been found in the space of an electrically charged mass-point (Reissner-Nordström's metric), with the difference being that there is a mass-defect due to both the gravitational and electromagnetic fields. The presence of an electromagnetic field in the mass of a particle was never considered in this fashion prior to the present study.

No mass-defect has been found in the rotating space of Gödel's metric, and in the space filled with a homogeneous distribution of ideal liquid and physical vacuum (Einstein's metric). This means that a mass-bearing particle does not achieve an add-on to its mass with the distance travelled in a Gödel universe or in an Einstein universe.

The other obtained solutions manifest a mass-defect of another sort than that in the case of the mass-point metric. Its magnitude increases with the distance travelled by the particle. Thus this mass-defect manifests itself at cosmologically large distances travelled by the particle. We therefore refer to it as the *cosmological mass-defect*.

According to the calculations presented in this study, cosmological mass-defect has been found in the space of Schwarzschild's metric of a sphere of incompressible liquid, in the space of a sphere filled with physical vacuum (de Sitter's metric), and in the deforming spaces of Friedmann's metric (empty or filled with ideal liquid and physical vacuum). In other words, a mass-bearing particle travelling in each of these spaces changes its mass according to the travelled distance.

The origin of this effect is the presence of gravitation, non-holonomity, and deformation of the space wherein the particle travels (if at least one of the factors is presented in the space): these are only three factors affecting the mass of a mass-bearing particle according to the scalar geodesic equation. In other words, a particle which travels in the field gains an additional mass due to the field's work accelerating the particle, or it loses its own mass due to the work against the field (depending on the condition in the particular space).

All these results have been obtained only due to the chronometrically invariant formalism, which has led us to the new method of deduction through integrating the scalar geodesic equation (equation of energy) of a mass-bearing particle.

Note that cosmological mass-defect — an add-on to the mass of a particle according to the travelled distance — has never been considered prior to the present study. It is, therefore, a new effect predicted due to the General Theory of Relativity.

A next step should logically be the calculation of the frequency shift of a photon according to the distance travelled by it. At first glance, this problem could be resolved very easily due to the similarity of the geodesic equations for mass-bearing particles and massless (light-like) particles (photons). However, this is not a trivial task. This is because massless particles travel in the isotropic space (home of the trajectories of light), which is strictly non-holonomic so that the lines of time meet the three-dimensional coordinate lines therein (hence the isotropic space rotates as a whole in each its point with the velocity of light). Therefore, all problems concerning massless (light-like) particles should be considered only by taking the strict non-holonomic condition of the isotropic space into account. I will focus on this problem, and on the calculation of the frequency shift of a photon according to the travelled distance, in the next paper (under preparation).

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