

# Unification of Interactions in Discrete Spacetime

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I assume that both spacetime and the internal symmetry space of the Standard Model (SM) of leptons and quarks are discrete. If lepton and quark states represent specific finite binary rotational subgroups of the SM gauge group, unification with gravitation is accomplished by combining finite subgroups of the Lorentz group  $SO(3,1)$  with the specific finite SM subgroups. The unique result is a particular finite subgroup of  $SO(9,1)$  in discrete 10-D spacetime related to  $E_8 \times E_8$  of superstring theory. A physical model of particles based upon the finite subgroups and the discrete geometry is proposed. Evidence for discreteness might be the appearance of a  $b'$  quark at about 80–100 GeV decaying via FCNC to a  $b$  quark plus a photon at the Large Hadron Collider.

## 1 Introduction

I consider both spacetime and the internal symmetry space of the Standard Model (SM) of leptons and quarks to be discrete instead of continuous. Using specific finite subgroups of the SM gauge group, a unique finite group in discrete 10-D spacetime unifies the fundamental interactions, including gravitation. This finite group is a special subgroup of the continuous group  $E_8 \times E_8$  that in superstring theory (also called M-theory) is considered to be the most likely group for unifying gravitation with the SM gauge group.

This unique result follows directly from two fundamental assumptions: (1) the internal symmetry space is discrete, requiring specific finite binary rotational subgroups of the SM gauge group to dictate the physical properties of the lepton and quark states, and (2) spacetime is discrete, and therefore its discrete symmetries correspond to finite subgroups of the Lorentz group. Presumably, this discreteness must occur as one approaches the Planck scale of about  $10^{-35}$  meters.

I suggest a particular physical model of fundamental fermions based upon these finite subgroups in the discrete geometry. Further evidence for this discreteness might be the appearance of a  $b'$  quark at about 80–100 GeV decaying via FCNC to a  $b$  quark plus a photon at the Large Hadron Collider.

## 2 Motivation

The Standard Model (SM) of leptons and quarks successfully describes their electromagnetic, weak and color interactions in terms of symmetries dictated by the  $SU(2)_L \times U(1)_Y \times SU(3)_C$  continuous gauge group. These fundamental fermions and their antiparticles are defined by their electroweak isospin states in two distinct but gauge equivalent unitary planes in an internal symmetry space “attached” at a space-

time point. Consequently, particle states and antiparticle states have opposite-signed physical properties but their masses are the same sign.

In an earlier 1994 paper [1] I discussed how the SM continuous gauge group could be acting like a “cover group” for its finite binary rotational subgroups, thereby hiding any important underlying discrete rotational symmetries of these fundamental particle states. From group theory, one knows that the continuous SM gauge group contains thousands of elements of finite order including, for example, all the elements of the finite binary rotational subgroups in their 3-dimensional and 4-dimensional representations. I showed that these subgroups were very important because they are connected to the  $j$ -invariant of elliptic modular functions from which one can predict the mass ratios for the lepton and quark states.

The mathematical properties of these finite subgroups of the SM dictate the same physical properties of the leptons and quarks as achieved by the SM. However, electroweak symmetry breaking to these specific finite binary rotational subgroups occurs without a Higgs particle. More importantly, some additional physical properties are dictated also, such as their mass ratios, why more than one generation is present, the important family relationships, and the dimensionalities of the particle states because they are no longer point particles.

The gravitational interaction is not included explicitly in the SM gauge group. However, because the finite binary rotational subgroup approach determined the lepton and quark mass ratios, one suspects that the gravitational interaction is included already in the discretized version of the gauge group. Or, equivalently, since mass/energy is the source of the gravitational interaction, the gravitational interaction arises from the discrete symmetries associated with the finite rotational subgroups.

Therefore, I make some conjectures. If leptons and quarks actually represent the specific discrete symmetries of the finite subgroups of the SM gauge group as proposed, the internal symmetry space may be discrete instead of being continuous. Going one step further, then not only the internal symmetry space might be discrete but also spacetime itself may be discrete, since gravitation determines the spacetime metric. Spacetime would appear to be continuous only at the low resolution scales of experimental apparatus such as the present particle colliders. Unification of the fundamental interactions then requires combining these finite groups mathematically.

### 3 4-D internal symmetry space?

I take the internal symmetry space of the SM to be discrete, but we need to know how many dimensions there are. Do we need two complex spatial dimensions for a unitary plane as suggested by  $SU(2)$ , or do we need three as suggested by the  $SU(3)$  symmetry of the color interaction, or do we need more?

The lepton and quark particle states are defined as electroweak isospin states by the electroweak part of the SM gauge group, with particles in the normal unitary plane  $C^2$  and antiparticles in the conjugate unitary plane  $C'^2$ . Photon,  $W^+$ ,  $W^-$ , and  $Z^0$  interactions of the electroweak  $SU(2)_L \times U(1)_Y$  gauge group rotate the two particle states (i. e., the two complex basis spinors in the unitary plane) into one another. For example,  $e^- + W^+ \rightarrow \nu_e$ . These electroweak rotations can be considered to occur also in an equivalent 4-dimensional real euclidean space  $R^4$  and in an equivalent quaternion space  $Q$ , both these spaces being useful for a better geometrical understanding of the SM.

The quark states are defined also by the color symmetries of  $SU(3)_C$ , i. e., each quark comes in one of three possible colors, red R, green G, or blue B, while the lepton states have no color charge. Normally, one would consider  $SU(3)_C$  operating in a space of three complex dimensions, or its equivalent six real dimensions. In fact,  $SU(3)_C$  can operate successfully in the smaller unitary plane  $C^2$ , because each  $SU(3)$  operation can be written as the product of three specific  $SU(2)$  operations [2]. An alternative geometrical explanation has the gluon operations of the color interaction rotate one color state into another in a 4-dimensional real space, as discussed in my 1994 article. Briefly, real 4-dimensional space  $R^4$  has four orthogonal coordinates  $(w, x, y, z)$ , and its 4-D rotations occur simultaneously in two orthogonal planes. There being only three distinct pairs of orthogonal planes,  $[wx, yz]$ ,  $[xy, zw]$ , and  $[yw, xz]$ , each color R, G, or B is assigned to a specific pair, thereby making color an exact geometrical symmetry. Consequently, the gluon operations of  $SU(3)_C$  occur in the 4-D real space  $R^4$  that is equivalent to the unitary plane. Detailed matrix operations confirm that

hadrons with quark-antiquark pairs, three quarks, or three antiquarks, are colorless combinations.

Therefore I take the internal symmetry space to be a discrete 4-dimensional real space because this space is the minimum dimensional space that allows the SM gauge group to operate completely. One does not need a larger space, e. g., a 6-dimensional real space, for its internal symmetry space.

### 4 Dimensions of spacetime?

I take physical spacetime to be 4-dimensional with its one time dimension. Spacetime is normally considered to be continuous and 4-dimensional, with three spatial dimensions and one time dimension. However, in the last two decades several approaches toward unifying all fundamental interactions have considered additional mathematical spatial dimensions and/or more time dimensions. For example, superstring theory [3] at the high energy regime, i. e., at the Planck scale, proposes 10 or 11 spacetime dimensions in its present mathematical formulation, including the one time dimension. These extra spatial dimensions may correspond to six or seven dimensions “curled up” into an internal symmetry space for defining fundamental particle states at each spacetime point in order to accommodate the SM in the low energy regime. The actual physical spacetime itself may still have three spatial dimensions and one time dimension.

I take 4-D spacetime to be discrete. We do not know whether spacetime is continuous or discrete. If the internal symmetry space is indeed discrete, then perhaps spacetime itself might be discrete also. Researchers in loop quantum gravity [4] at the Planck scale divide spacetime into discrete subunits, considering a discrete 4-D spacetime with its discrete Lorentz transformations to be a viable approach.

The goal now is to combine the finite subgroups of the gauge group of the SM and the finite group of discrete Lorentz boosts and discrete spacetime rotations into one unified group. All four known fundamental interactions would be unified. Although many unification schemes for the fundamental interactions have been attempted over the past three decades utilizing continuous groups, I believe this attempt is the first one that combines finite groups. Mathematically, the result must be unique, otherwise different fundamental laws could exist in different parts of the universe.

### 5 Discrete internal space

The most important finite symmetry groups in the 4-D discrete internal symmetry space are the 3-D binary rotational subgroups  $[3, 3, 2]$ ,  $[4, 3, 2]$ , and  $[5, 3, 2]$  of the SM gauge group because they are the symmetry groups I have assigned to the three lepton families. They contain discrete rotations and inversions and operate in the 3-D subspace  $R^3$  of  $R^4$  and  $C^2$ .

Being subgroups of  $SU(2)_L \times U(1)_Y$ , they have group operations represented by  $2 \times 2$  unitary matrices or, equivalently, by unit quaternions. Quaternions provide the more obvious geometrical connection [5], because quaternions perform the dual role of being a group operation and of being a vector in  $R^3$  and in  $R^4$ . One can think visually about the 3-D group rotations and inversions for these three subgroups as quaternions operating on the Platonic solids, with the same quaternions also defining the vertices of regular geometrical objects in  $R^4$ .

The two mathematical entities, the unit quaternion  $\mathbf{q}$  and the  $SU(2)$  matrix, are related by

$$\mathbf{q} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \iff \begin{pmatrix} w + iz & x + iy \\ -x + iy & w - iz \end{pmatrix} \quad (1)$$

where the  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  are unit imaginaries, their coefficients are real, and  $w^2 + x^2 + y^2 + z^2 = 1$ . The conjugate quaternion  $\mathbf{q}' = w - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$  and its corresponding matrix would represent the same group operation in the conjugate unitary plane for the antiparticles. Recall that Clifford algebra and Bott periodicity dictate that only  $R^4$ ,  $R^8$ , and other real spaces  $R^n$  with dimensions divisible by four have two equivalent conjugate spaces, the specific mathematical property that accommodates both particle states and antiparticle states. The group  $U(1)_Y$  for weak hypercharge  $Y$  then reduces the symmetry to being gauge equivalent so that particles and antiparticles have the same positive mass.

One might expect that we need to analyze each of the three binary rotational subgroups separately when the discrete internal symmetry space is combined with discrete spacetime. Fortunately, the largest binary rotational group [5, 3, 2] of icosahedral symmetries can accommodate the two other groups, and a discussion of its 120 quaternion operations is all inclusive mathematically. The elements of this icosahedral group, rotations and inversions, can be represented by the appropriate unit quaternions.

The direct connection between the 3-D and 4-D spaces is realized when one equates the 120 group operations on the regular icosahedron (3, 5) to the vectors for the 120 vertices of the 600-cell hypericosahedron (3, 3, 5) in 4-D space in a particular way. These operations of the binary icosahedral group [5, 3, 2] and the vertices of the hypericosahedron are defined by 120 special unit quaternions  $q_i$  known as isosians [6], which have the mathematical form

$$q_i = \left( e_1 + e_2\sqrt{5} \right) + \left( e_3 + e_4\sqrt{5} \right) \mathbf{i} + \left( e_5 + e_6\sqrt{5} \right) \mathbf{j} + \left( e_7 + e_8\sqrt{5} \right) \mathbf{k}, \quad (2)$$

where the eight  $e_j$  are special rational numbers. Specifically, the 120 icosians are obtained by permutations of

$$\begin{aligned} & (\pm 1, 0, 0, 0), (\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2), \\ & (0, \pm 1/2, \pm g/2, \pm G/2), \end{aligned} \quad (3)$$

where  $g = G^{-1} = G - 1 = (-1 + \sqrt{5})/2$ . Notice that in each pair, such as  $(e_3 + e_4\sqrt{5})$ , only one of the  $e_j$  is nonzero, reminding us that the hypericosahedron is really a 4-D object even though we can now define this object in terms of icosians that are expressed in the much larger  $R^8$  euclidean real space.

So the quaternion's dual role allows us to identify the 120 group operations of the icosahedron with the 120 vertices of the hypericosahedron expressed both in  $R^4$  and in  $R^8$ , essentially telescoping from 3-D rotational operations all the way to their representations in an 8-D space. These special 120 icosians are to be considered as special octonions, 8-tuples of rational numbers which, with respect to a particular norm, form part of a special lattice in  $R^8$ .

Now consider the two other subgroups. The 24 quaternions of the binary tetrahedral group [3, 3, 2] are contained already in the above 120 icosians. So we are left with accommodating the binary octahedral group [4, 3, 2] into the same icosian format. We need 48 special quaternions for its 48 operations, the 24 quaternions defining the vertices of the 4-D object known as the 24-cell contained already in the hypericosahedron above and another 24 quaternions for the reciprocal 24-cell. The 120 unit quaternions reciprocal to the ones above will meet this requirement as well as define an equivalent set for the reciprocal hypericosahedron, and this second set of 120 octonions also forms part of a special lattice in  $R^8$ . Together, these two lattice parts of 120 icosians in each combine to form the 240 octonions of the famous  $E_8$  lattice in  $R^8$ , well known for being the densest lattice packing of spheres in 8-D.

Recall that the three binary rotation groups above are assigned to the lepton families because, as subgroups of the SM gauge group, they predict the correct physical properties of the lepton states, including the correct mass ratios. Therefore, the lepton states as I have defined them span only the 3-D real subspace  $R^3$  of the unitary plane. That is why leptons are color neutral and do not participate in the color interaction, a physical property that requires the ability to undergo complete 4-D rotations.

So how do quark states fit into the icosian picture? I have the quark states in the SM spanning the whole 4-D real space, i. e., the whole unitary plane, because they are the basis states of the 4-D finite binary rotational subgroups of the SM gauge group. But free quarks in spacetime do not exist because they are confined according to QCD, forming the colorless quark-antiquark, three-quark, or three-antiquark combinations called hadrons. Mathematically, these colorless hadron states span the 3-D subspace only, so their resultant discrete symmetry group must be isomorphic to one of the three binary rotational subgroups we have just considered. Consequently, the icosians enumerated above account for all the lepton states and for all the quark states in their allowed hadronic combinations.

## 6 Discrete spacetime

Linear transformations in discrete spacetime are the discrete rotations and the discrete Lorentz boosts. Before considering these discrete transformations, however, I discuss the continuous transformations of the “heavenly sphere” as a useful mathematical construct before reducing the symmetry to discrete transformations in a discrete spacetime.

The continuous Lorentz group  $SO(3,1)$  contains all the rotations and Lorentz boosts, both continuous and discrete, for the 4-D continuous spacetime with the Minkowski metric. Its operations are quaternions because there exists the isomorphism

$$SO(3, 1) = PSL(2, \mathbb{C}). \quad (4)$$

The group  $PSL(2, \mathbb{C})$  consists of unit quaternions and is the quotient group  $SL(2, \mathbb{C})/Z$  formed by its center  $Z$ , those elements of  $SL(2, \mathbb{C})$  which commute with all the rest of the group. Its  $2 \times 2$  matrix representation has complex numbers as entries.

The continuous Lorentz transformations (including the spatial rotations) operate on the “heavenly sphere” [7], i. e., the famous Riemann sphere formed by augmenting the complex plane  $\mathbb{C}$  by the “point at infinity”. The Riemann sphere is also the space of states of a spin-1/2 particle. For the Lorentz transformations in spacetime, if you are located at the center of this “heavenly sphere” so that the light rays from stars overhead each pass through unique points on a unit celestial sphere surrounding you, then the Lorentz boost is a conformal transformation of the star locations. The constellations will look distorted because the apparent lengths of the lines connecting the stars will change but the angles between these connecting lines will remain the same.

These conformal transformations are called fractional linear transformations, or Möbius transformations, of the Riemann sphere, expressed by the general form [8]

$$w \mapsto \frac{\alpha w + \beta}{\gamma w + \delta}, \quad (5)$$

with  $\alpha, \beta, \gamma$ , and  $\delta$  complex, and  $\alpha\delta - \beta\gamma \neq 0$ . The  $2 \times 2$  matrix representation for transformation of a spinor  $\nu$  as the map  $\nu \mapsto M\nu$  is

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \quad (6)$$

Thus,  $M$  is the spinor representation of the Lorentz transformation.  $M$  acts on a vector  $A = \nu\nu^\dagger$  via  $A \mapsto MAM^\dagger$  [9]. All these relationships are tied together by the group isomorphisms in continuous 4-D spacetime

$$SO(3, 1) = \text{Möbius group} = PSL(2, \mathbb{C}). \quad (7)$$

Discrete spacetime has discrete Lorentz transformations, not continuous ones. These discrete rotations and discrete Lorentz boosts are contained already in  $SO(3,1)$ , and they

tessellate the Riemann sphere. That is, they form regular polygons on its surface that correspond to the discrete symmetries of the binary tetrahedral, binary octahedral, and binary icosahedral rotation groups [3, 3, 2], [4, 3, 2], and [5, 3, 2], the same groups I used in the internal symmetry space for the discrete symmetries. Therefore, the 240 quaternions defined previously are required also for the discrete rotations and discrete Lorentz boosts in the discrete 4-D spacetime. Again, there are the same 240 icosian connections to octonions in  $R^8$  to form a second  $E_8$  lattice.

Thus, the Lorentz group  $SO(3,1)$  with its linear transformations in a continuous 4-D spacetime, when reduced to its discrete transformations in a 4-D discrete spacetime, is connected mathematically by icosians to the  $E_8$  lattice in  $R^8$ , telescoping the transformations from a smaller discrete spacetime to a larger one. Hence all linear transformations for the particles in a 4-D discrete spacetime have become represented by 240 discrete transformations in the 8-D discrete spacetime.

## 7 Resultant spacetime

The discrete transformations in the 4-D discrete internal symmetry space and in the 4-D discrete spacetime are each represented by an  $E_8$  lattice in the 8-D space  $R^8$ . The finite group of the discrete symmetries of the  $E_8$  lattice is the Weyl group  $E_8$ , not to be confused with the continuous exceptional Lie group  $E_8$ . Thus, the Weyl  $E_8$  is a finite subgroup of  $SO(8)$ , the continuous group of all rotations of the unit sphere in  $R^8$  with determinant unity. In this section I combine the two Weyl  $E_8$  groups to form a bigger group that operates in a discrete spacetime, and then in the next section I suggest a simple physical model for fundamental fermions that would fit the geometry.

I have now two sets of 240 icosians each forming  $E_8$  lattices in  $R^8$ , each obeying the symmetry operations of the finite group Weyl  $E_8$ . Each finite group of octonions acts as rotations and as vectors in  $R^8$ . I identify their direct product as the elements of a discrete subgroup of the continuous group  $PSL(2, \mathbb{O})$ , where  $\mathbb{O}$  represents all the unit octonions. That is, if all the unit octonions in each were present, not just the subset of unit octonions that form the  $E_8$  lattice, their direct product group would be the continuous group of  $2 \times 2$  matrices in which all matrix entries are unit octonions. So the spinors in  $R^8$  are octonions.

The 8-D result is analogous to the 4-D result but different. Recall that in the 4-D case, one has  $PSL(2, \mathbb{C})$ , the group of  $2 \times 2$  matrices with complex numbers as entries, with  $PSL(2, \mathbb{C}) = SO(3,1)$ , the Lorentz group in 4-D spacetime. Here in 8-D one has a surprise, for the final combined spacetime is bigger, being isomorphic to a 10-dimensional spacetime instead of 8-dimensional spacetime because

$$PSL(2, \mathbb{O}) = SO(9, 1), \quad (8)$$

the Lorentz group in 10-D spacetime.

Applied to the discrete case, the combined group is the finite subgroup

$$\text{finite PSL}(2, \mathbb{O}) = \text{finite SO}(9, 1), \quad (9)$$

that is, the finite Lorentz group in discrete 10-D spacetime. The same results, expressed in terms of the direct product of Weyl  $E_8$  groups, is

$$\text{Weyl } E_8 \times \text{Weyl } E_8 = \text{“Weyl” SO}(9, 1), \quad (10)$$

where “Weyl”  $\text{SO}(9,1)$  is defined by the direct product on the left and is a finite subgroup of  $\text{SO}(9,1)$ .

Working in reverse, the discrete 10-D spacetime divides into two parts as a 4-D discrete spacetime plus a 4-D discrete internal symmetry space. There is a surprise in this result: combining a discrete 4-D internal symmetry space with a discrete 4-D spacetime creates a discrete 10-D spacetime, not a discrete 8-D spacetime. Therefore, a continuous 10-D spacetime, when “discretized”, is not required to partition into a 4-D spacetime plus a 6-D “curled up” space as proposed in superstring theory.

## 8 A physical particle model

In the 1994 paper I proposed originally that leptons have the symmetries of the 3-D regular polyhedral groups and that quarks have the symmetries of the 4-D regular polytope groups. Now that I have combined the discrete 4-D internal symmetry space with a discrete 4-D spacetime to achieve mathematically a discrete 10-D spacetime, the fundamental question arises: Are the leptons and quarks really 3-D and 4-D objects physically, or are they something else, perhaps 8-D or 10-D objects?

In order to answer this question I need to formulate a reasonable physical model of fundamental particles in this discrete spacetime environment. The simplest mathematical viewpoint is that discrete spacetime is composed of identical entities, call them nodes, which have no *measurable* physical properties until they collectively distort spacetime to form a fundamental particle such as the electron, for example. The collection of nodes and its distortion of the surrounding spacetime exhibit the discrete symmetry of the appropriate finite binary rotation group for the specific particle. For example, the electron family has the discrete symmetry of the binary tetrahedral group and the electron is one of its two possible orthogonal basis states. So the distortion for the collection of nodes called the electron will exhibit the discrete symmetries of its  $[3, 3, 2]$  group as all of its physical properties emerge for this specific collection and did not exist beforehand. The positron forms in the conjugate space.

One can begin with a regular lattice of nodes in both the normal unitary plane and in its conjugate unitary plane, or one can consider the equivalent  $R^4$  spaces, and then

imagine that a spacetime distortion appears in both to form a particle-antiparticle pair. Mathematically, one begins with an isotropic vector, also called a zero length vector, which is orthogonal to itself, that gets divided into two unit spinors corresponding to the creation of the particle-antiparticle pair. No conservation laws are violated because their quantum numbers are opposite and the sum of the total mass energy plus their total potential energy is zero. The spacetime distortion that is the particle and its “field” mathematically brings the nodes closer together locally with a corresponding adjustment to the node spacing all the way out to infinite distance, all the while keeping the appropriate discrete rotational symmetry intact. The gravitational interaction associated with this discrete symmetry therefore extends to infinite distance.

This model of particle geometry must treat leptons as 3-D objects and quarks as 4-D objects in a discrete 4-D spacetime. We know that there are no isolated quarks, for they immediately form 3-D objects called hadrons. These lepton states and hadron states are described by quaternions of the form  $w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ , so these 3-D objects “live” in the three imaginary dimensions, and the 4th dimension can be called time. Therefore, leptons and hadrons each experience the “passage of time”, while individual quarks do not have this characteristic until they form hadrons in the 3-D subspace.

If this physical model is a reasonable approximation to describing the world of fundamental particles, why are superstring researchers working in 10-dimensions or more? Because one desires a single symmetry group that includes both the group of spacetime transformations of particles and the group of internal symmetries for the particle interactions. At the Planck scale, if one has a continuous group, then the smallest dimensional continuous spacetime one can use is 10-D in order to have a viable Lagrangian. Reducing this 10-D spacetime to the low energy regime of the SM in 4-D spacetime, the 10-D continuous spacetime has been postulated to divide into 4-D spacetime plus an additional 6-dimensional “curled up” space in which to accommodate the SM. In M-theory, one may be considering an 11-D spacetime dividing into a 4-D spacetime plus a 7-D “curled up” space. But this approach using continuous groups to connect back to the SM has proven difficult, although some significant advances have been achieved.

The analysis presented above for combining the two finite Weyl  $E_8$  groups shows that the combined group operates in 10-D discrete spacetime with all the group operations being discrete. The particles are 3-D objects “traveling” in spacetime. No separate “curled up” space is required at the low energy limit corresponding to a distance scale of about  $10^{-23}$  meters or larger. The discreteness at the Planck scale and the “hidden” discreteness postulated for all larger distance scales is the mathematical feature that permits the direct unique connection through icosians from the high energy world to the familiar lower energy world of the SM.

## 9 Mathematical connections

The mathematical connections of these binary polyhedral groups to number theory, geometry, and algebra are too numerous to list and discuss in this short article. In fact, according to B. Kostant [10], if one were to choose groups in mathematics upon which to construct the symmetries of the universe, one couldn't choose a better set, for "... in a very profound way, the finite groups of symmetries in 3-space 'see' the simple Lie groups (and hence literally Lie theory) in all dimensions." Therefore, I provide a brief survey of a few important connections here and will discuss them in more detail in future articles.

Geometrical connections are important for these groups. The continuous group  $PSL(2, \mathbb{C})$  defines a torus, as does  $PSL(2, \mathbb{O})$ . In the discrete environment, finite  $PSL(2, \mathbb{C})$  and finite  $PSL(2, \mathbb{O})$  have special symmetry points on each torus corresponding to the elements of the finite binary polyhedral groups. An important mathematical property of the binary polyhedral groups is their connection to elliptic modular functions, the doubly periodic functions, and their famous  $j$ -invariant function, which has integer coefficients in its series expansion related to the largest of the finite simple groups called the Monster.

The binary tetrahedral, octahedral and icosahedral rotation groups are the finite groups of Möbius transformations  $PSL(2, Z_3)$ ,  $PSL(2, Z_4)$ , and  $PSL(2, Z_5)$ , respectively, where  $Z_n$  denotes integers mod  $(n)$ .  $PSL(2, Z_n)$  is often called the modular group  $\Gamma(n)$ .  $PSL(2, Z_n) = SL(2, Z_n) / \{\pm I\}$ , so these three binary polyhedral groups (along with the cyclic and dihedral groups) are the finite modular subgroups of  $PSL(2, \mathbb{C})$  and are also discrete subgroups of  $PSL(2, \mathbb{R})$ .  $PSL(2, Z_n)$  is simple in only three cases:  $n = 5, 7, 11$ . And these three cases are the Platonic groups again:  $A_5$  and its subgroup  $A_4$ ,  $S_4$ , and  $A_5$ , respectively [11].

An important mathematical property for physics is that our binary polyhedral groups, the  $\Gamma(n)$ , are generated by the two transformations

$$X : \tau \mapsto -1/\tau \quad Y : \tau \mapsto \tau + 1, \quad (11)$$

with  $\tau$  being the lattice parameter for the plane associated with forming the tessellations of the toroidal Riemann surface. The  $j$ -invariant function  $j(\tau)$  of elliptic modular functions exhibits this transformation behavior. Consequently, functions describing the physical properties of the fundamental leptons and quarks will exhibit these same transformation properties. So here is where the duality theorems of M-theory, such as the  $S$  duality relating the theory at physical coupling  $g$  to coupling at  $1/g$ , arise naturally from mathematical properties of the finite binary polyhedral groups.

One can show also that octonions and the triality connection for spinors and vectors in  $R^8$  are related to the fundamental interactions. In 8-D, the fundamental matrix representations both for left- and right-handed spinors and for

vectors are the same dimension,  $8 \times 8$  [12], leading to many interesting mathematical properties. For example, an electron represented by a left-handed octonionic spinor interacting with a  $W^+$  boson represented by an octonionic vector becomes an electron neutrino, again an octonionic spinor. Geometrically, this interaction looks like three  $E_8$  lattices combining momentarily to form the famous 24-dimensional Leech lattice!

By using a discrete spacetime, we have begun to suspect that Nature has established a universe based upon fundamental mathematics that dictates unique fundamental physics principles. Moreover, one might expect that all physical constants will be shown to arise from fundamental mathematical relationships, dictating one universe with unique constant values for a unique set of fundamental laws.

## 10 Experimental tests

There is no direct test yet devised for discrete spacetime. However, my discrete internal symmetry space approach dictates a fourth quark family with a  $b'$  quark state at about 80 GeV and a  $t'$  quark at about 2600 GeV. The production of this  $b'$  quark with the detection of its decay to a  $b$  quark and a high energy photon seems at present to be the only attainable empirical test for discreteness. Its appearance in collider decays would be an enormously important event in particle physics, strongly suggesting that the internal symmetry space and its "surrounding" spacetime are discrete.

However, the  $b'$  quark has remained hidden among the collision debris at Fermilab because its flavor changing neutral current (FCNC) decay channel has a very low probability compared to all the other particle decays in this energy regime. This  $b'$  quark decay may even be confused with the decay of the Higgs boson, should such a particle exist, until all the quantum numbers are established. The  $t'$  quark at around 2600 GeV has too great a mass to have been produced directly at Fermilab.

I expect the production of  $b'$  quarks at the Large Hadron Collider in a few years to be the acid test for discreteness and to verify the close connection of fundamental physics to the mathematical properties of the finite simple groups.

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## References

1. Potter F. Geometrical basis for the Standard Model. *International Journal of Theoretical Physics*, 1994, v.33(2), 279–305.

2. Rowe D.J., Sanders B.C., de Guise H. Representations of the Weyl group and Wigner functions for SU(3). *Journal of Mathematical Physics*, 1999, v. 40(7), 3604–3615.
  3. Schwarz J.H. Update on String Theory. arXiv: astro-ph/0304507.
  4. Urrutia L.F. Flat space modified particle dynamics induced by Loop Quantum Gravity. arXiv: hep-ph/0402271.
  5. Coxeter H.S.M. Regular complex polytopes. Cambridge University Press, Cambridge, 1974, pp. 89–97.
  6. Conway J.H., Sloane N.J.A. Sphere packings, lattices and groups. 3rd ed. Springer-Verlag, New York, 1998.
  7. Penrose R., Rindler W. Spinors and space-time, Volume 1, Reprint edition. Cambridge University Press, Cambridge, 1987, pp. 26–28.
  8. Jones, G., Singerman, D. Complex functions: an algebraic and geometric viewpoint. Cambridge University Press, Cambridge, 1987, pp. 17–53.
  9. Manogue C.A., Dray T. Octonionic Möbius transformations. *Modern Physics Letters v.A14*, 1999, 1243–1256. arXiv: math-ph/9905024.
  10. Kostant B. In *Asterisque*, (Proceedings of the Conference “Homage to Elie Cartan”, Lyons), 1984, p. 13.
  11. Kostant B. The graph of the truncated icosahedron and the last letter of Galois. *Notices of the American Mathematical Society*, 1995, v. 42, 959–968.
  12. Baez J. The octonions. arXiv: math.RA/0105155.
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