On Area Coordinates and Quantum Mechanics in Yang's Noncommutative Spacetime with a Lower and Upper Scale

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We explore Yang's Noncommutative space-time algebra (involving *two* length scales) within the context of QM defined in Noncommutative spacetimes and the holographic area-coordinates algebra in Clifford spaces. Casimir invariant wave equations corresponding to Noncommutative coordinates and momenta in d-dimensions can be recast in terms of *ordinary* QM wave equations in $d+2$-dimensions. It is conjectured that QM over Noncommutative spacetimes (Noncommutative QM) may be described by ordinary QM in *higher* dimensions. Novel Moyal-Yang-Fedosov-Kontsevich star products deformations of the Noncommutative Poisson Brackets are employed to construct star product deformations of scalar field theories. Finally, generalizations of the Dirac-Konstant and Klein-Gordon-like equations relevant to the physics of D-branes and Matrix Models are presented.

1 Introduction

Yang's noncommutative space time algebra [1] is a generalization of the Snyder algebra [2] (where now both coordinates and momenta are not commuting) that has received more attention recently, see for example [3] and references therein. In particular, Noncommutative p-brane actions, for even $p+1 = 2n$-dimensional world-volumes, were written explicitly [15] in terms of the *novel* Moyal-Yang (Fedosov-Kontsevich) star product deformations [11, 12] of the Noncommutative Nambu Poisson Brackets (NCNPB) that are associated with the *noncommuting* world-volume coordinates q^A, p^A for $A = 1, 2, 3, \ldots n$. The latter noncommuting coordinates obey the noncommutative Yang algebra with an ultraviolet L_P (Planck) scale and infrared (R) scale cutoff. It was shown why the novel p-brane actions in the "classical" limit $\hbar_{\text{eff}} = \hbar L_P/R \rightarrow 0$ still acquire nontrivial noncommutative *corrections* that differ from ordinary p-brane actions. Super p-branes actions in the light-cone gauge are also amenable to Moyal-Yang star product deformations as well due to the fact that p-branes moving in flat spacetime backgrounds, in the light-cone gauge, can be recast as gauge theories of volume-preserving diffeomorphisms. The most general construction of noncommutative super p-branes actions based on non (anti) commuting superspaces and quantum group methods remains an open problem.

The purpose of this work is to explore further the consequences of Yang's Noncommutative spacetime algebra within the context of QM in Noncommutative spacetimes and the holographic area-coordinates algebra in Clifford spaces [14]. In section 2 we study the interplay among Yang's Noncommutative spacetime algebra and the former area-coordinates algebra in Clifford spaces. In section 3 we show how Casimir invariant wave equations corresponding to Noncommutative coordinates and momenta in D-dimensions, can be recast in terms of ordinary QM wave equations in $D+2$-dimensions. In particular, we shall present explicit solutions of the D'Alambertian operator in the *bulk* of AdS spaces and explain its correspondence with the Casimir invariant wave equations associated with the Yang's Noncommutative spacetime algebra at the projective *boundary* of the conformally compactified AdS spacetime. We conjecture that QM over Noncommutative spacetimes (Noncommutative QM) may be described by ordinary QM in *higher* dimensions.

In section 4 we recur to the *novel* Moyal-Yang (Fedosov-Kontsevich) star products [11, 12] deformations of the Noncommutative Poisson Brackets to construct Moyal-Yang star product deformations of scalar field theories. The role of star products in the construction of p-branes actions from the large N limit of $SU(N)$ Yang-Mills can be found in [6] and in the Self-Dual Gravity/$SU(\infty)$ Self Dual Yang-Mills relation in [7, 8, 9, 10]. Finally, in the conclusion 5, we present the generalizations of the Dirac-Konstant equations (and their "square" Klein-Gordon type equations) that are relevant to the incorporation of fermions and the physics of D-branes and Matrix Models.

2 Noncommutative Yang's spacetime algebra in terms of area-coordinates in Clifford spaces

The main result of this section is that there is a *subalgebra* of the C-space operator-valued coordinates [13] which is *isomorphic* to the Noncommutative Yang's spacetime algebra [1, 3]. This, in conjunction to the discrete spectrum of angular momentum, leads to the discrete area quantization in multiples of Planck areas. Namely, the $4D$ Yang's Noncommutative space-time algebra [3] (written in terms of $8D$ phase-space coordinates) is isomorphic to the 15-dimensional *subalgebra* of the C-space operator-valued coordinates associated with the *holographic areas* of C-space. This connection between Yang's algebra and the $6D$ Clifford algebra is possible because the $8D$ phase-space coordinates x^μ, p^μ (associated to a $4D$ spacetime) have a one-to-one correspondence to the $\hat{X}^{\mu 5}$; $\hat{X}^{\mu 6}$ holographic area-coordinates of the C-space (corresponding to the $6D$ Clifford algebra). Furhermore, Tanaka [3] has shown that the Yang's algebra [1] (with 15 generators) is related to the $4D$ conformal algebra (15 generators) which in turn is isomorphic to a subalgebra of the $4D$ Clifford algebra because it is known that the 15 generators of the $4D$ conformal algebra $SO(4,2)$ can be explicitly realized in terms of the $4D$ Clifford algebra as shown in [13].

The correspondence between the holographic area coordinates $X^{AB} \leftrightarrow \lambda^2 \Sigma^{AB}$ and the angular momentum variables when $A, B = 1, 2, 3, \dots 6$ yields an isomorphism between the holographic area coordinates algebra in Clifford spaces [14] and the noncommutative Yang's spacetime algebra in $D = 4$. The scale λ is the ultraviolet lower Planck scale. We begin by writing the exchange algebra between the position and momentum coordinates encapsulated by the commutator

$$\begin{gathered} \left[\hat{X}^{\mu 6}, \hat{X}^{56}\right] = -i\lambda^2 \eta^{66} \hat{X}^{\mu 5} \;\leftrightarrow \\ \left[\frac{\lambda^2 R}{\hbar}\,\hat{p}^\mu,\ \lambda^2 \Sigma^{56}\right] = -i\lambda^2 \eta^{66} \lambda \hat{x}^\mu \end{gathered} \tag{2.1}$$

from which we can deduce that

$$\left[\hat{p}^\mu,\ \Sigma^{56}\right] = -i\eta^{66} \frac{\hbar}{\lambda R}\,\hat{x}^\mu, \tag{2.2}$$

hence, after using the definition $\mathcal{N} = (\lambda/R)\,\Sigma^{56}$, where R is the infrared upper scale, one has the exchange algebra commutator of p^μ and $\mathcal{N}$ of the Yang's spacetime algebra given by

$$[\hat{p}^\mu, \mathcal{N}] = -i\eta^{66} \frac{\hbar}{R^2}\,\hat{x}^\mu. \tag{2.3}$$

From the commutator

$$\begin{gathered} \left[\hat{X}^{\mu 5}, \hat{X}^{56}\right] = -\left[\hat{X}^{\mu 5}, \hat{X}^{65}\right] = i\eta^{55} \lambda^2 \hat{X}^{\mu 6} \;\leftrightarrow \\ \left[\lambda \hat{x}^\mu, \lambda^2 \Sigma^{56}\right] = i\eta^{55} \lambda^2 \lambda^2 \frac{R}{\hbar}\,\hat{p}^\mu \end{gathered} \tag{2.4}$$

we can deduce that

$$\left[\hat{x}^\mu, \Sigma^{56}\right] = i\eta^{55} \frac{\lambda R}{\hbar}\,\hat{p}^\mu \tag{2.5}$$

and after using the definition $\mathcal{N} = (\lambda/R)\,\Sigma^{56}$ one has the exchange algebra commutator of x^μ and $\mathcal{N}$ of the Yang's spacetime algebra

$$[\hat{x}^\mu, \mathcal{N}] = i\eta^{55} \frac{\lambda^2}{\hbar}\,\hat{p}^\mu. \tag{2.6}$$

The other relevant holographic area-coordinates commutators in C-space are

$$\left[\hat{X}^{\mu 5}, \hat{X}^{\nu 5}\right] = -i\eta^{55} \lambda^2 \hat{X}^{\mu\nu} \leftrightarrow [\hat{x}^\mu, \hat{x}^\nu] = -i\eta^{55} \lambda^2 \Sigma^{\mu\nu} \tag{2.7}$$

that yield the noncommuting coordinates algebra after having used the representation of the C-space operator holographic area-coordinates

$$i\hat{X}^{\mu\nu} \leftrightarrow i\lambda^2 \frac{1}{\hbar} \mathcal{M}^{\mu\nu} = i\lambda^2 \Sigma^{\mu\nu}, \quad i\hat{X}^{56} \leftrightarrow i\lambda^2 \Sigma^{56}, \tag{2.8}$$

where we appropriately introduced the Planck scale λ as one should to match units. From the correspondence

$$\hat{p}^\mu = \frac{\hbar}{R}\,\Sigma^{\mu 6} \leftrightarrow \frac{\hbar}{R} \frac{1}{\lambda^2}\,\hat{X}^{\mu 6} \tag{2.9}$$

one can obtain nonvanishing momentum commutator

$$\left[\hat{X}^{\mu 6}, \hat{X}^{\nu 6}\right] = -i\eta^{66} \lambda^2 \hat{X}^{\mu\nu} \leftrightarrow [\hat{p}^\mu, \hat{p}^\nu] = -i\eta^{66} \frac{\hbar^2}{R^2} \Sigma^{\mu\nu}. \tag{2.10}$$

The signatures for AdS_5 space are $\eta^{55} = +1$; $\eta^{66} = -1$ and for the *Euclideanized* AdS_5 space are $\eta^{55} = +1$ and $\eta^{66} = +1$. Yang's space-time algebra corresponds to the latter case. Finally, the *modified* Heisenberg algebra can be read from the following C-space commutators

$$\begin{gathered} \left[\hat{X}^{\mu 5}, \hat{X}^{\nu 6}\right] = i\eta^{\mu\nu} \lambda^2 \hat{X}^{56} \;\leftrightarrow \\ [\hat{x}^\mu, \hat{p}^\mu] = i\hbar \eta^{\mu\nu} \frac{\lambda}{R}\,\Sigma^{56} = i\hbar \eta^{\mu\nu} \mathcal{N}. \end{gathered} \tag{2.11}$$

Eqs-(2.1–2.11) are the defining relations of Yang's Noncommutative $4D$ spacetime algebra [1] involving the $8D$ phase-space variables. These commutators obey the Jacobi identities. There are other commutation relations like $[\mathcal{M}^{\mu\nu}, x^\rho], \dots$ that we did not write down. These are just the well known rotations (boosts) of the coordinates and momenta.

When $\lambda \to 0$ and $R \to \infty$ one recovers the ordinary *commutative* spacetime algebra. The Snyder algebra [2] is recovered by setting $R \to \infty$ while leaving λ intact. To recover the ordinary Weyl-Heisenberg algebra is more subtle. Tanaka [3] has shown the the *spectrum* of the operator $\mathcal{N} = (\lambda/R)\,\Sigma^{56}$ is discrete given by $n(\lambda/R)$. This is not suprising since the angular momentum generator $\mathcal{M}^{56}$ associated with the *Euclideanized* AdS_5 space is a rotation in the now compact $x^5 - x^6$ directions. This is not the case in AdS_5 space since $\eta^{66} = -1$ and this timelike direction is no longer compact. Rotations involving timelike directions are equivalent to noncompact boosts with a continuous spectrum.

In order to recover the standard Weyl-Heisenberg algebra from Yang's Noncommutative spacetime algebra, and the standard uncertainty relations $\Delta x \Delta p \geqslant \hbar$ with the ordinary $\hbar$ term, rather than the $n\hbar$ term, one needs to take the limit $n \to \infty$ limit in such a way that the net combination of $n\frac{\lambda}{R} \to 1$. This can be attained when one takes the *double* scaling limit of the quantities as follows

$$\begin{gathered} \lambda \to 0, \quad R \to \infty\,, \quad \lambda R \to L^2, \\ \lim_{n \to \infty} n\,\frac{\lambda}{R} = n\,\frac{\lambda^2}{\lambda R} = \frac{n\lambda^2}{L^2} \to 1. \end{gathered} \tag{2.12}$$

From eq-(2.12) one learns then that

$$n\lambda^2 = \lambda R = L^2. \tag{2.13}$$

The spectrum n corresponds to the quantization of the angular momentum operator in the $x^5 - x^6$ direction (after embedding the $5D$ hyperboloid of throat size R onto $6D$). Tanaka [3] has shown why there is a *discrete spectra* for the *spatial* coordinates and *spatial* momenta in Yang's spacetime algebra that yields a *minimum* length λ (ultraviolet cutoff in energy) and a minimum momentum $p = \hbar/R$ (maximal length R, infrared cutoff). The energy and temporal coordinates had a continous spectrum.

The physical interpretation of the double-scaling limit of eq-(2.12) is that the the area $L^2 = \lambda R$ becomes now *quantized* in units of the Planck area λ^2 as $L^2 = n\lambda^2$. Thus the quantization of the area (via the double scaling limit) $L^2 = \lambda R = n\lambda^2$ is a result of the *discrete* angular momentum spectrum in the $x^5 - x^6$ directions of the Yang's Noncommutative spacetime algebra when it is realized by (angular momentum) differential operators acting on the *Euclideanized* AdS_5 space (two branches of a $5D$ hyperboloid embedded in $6D$). A general interplay between quantum of areas and quantum of angular momentum, for arbitrary values of spin, in terms of the square root of the Casimir $\mathbf{A} \sim \lambda^2 \sqrt{j\,(j+1)}$, has been obtained a while ago in Loop Quantum Gravity by using spin-networks techniques and highly technical area-operator regularization procedures [4].

The advantage of this work is that we have arrived at similar (not identical) area-quantization conclusions in terms of minimal Planck areas and a discrete angular momentum spectrum n via the double scaling limit based on Clifford algebraic methods (C-space holographic area-coordinates). This is not surprising since the norm-squared of the holographic Area operator has a correspondence with the quadratic Casimir $\Sigma_{AB}\Sigma^{AB}$ of the conformal algebra $SO(4,2)$ ($SO(5,1)$ in the Euclideanized AdS_5 case). This quadratic Casimir must not be confused with the $SU(2)$ Casimir J^2 with eigenvalues $j\,(j+1)$. Hence, the correspondence given by eqs-(2.3–2.8) gives $\mathbf{A}^2 \leftrightarrow \lambda^4 \Sigma_{AB}\Sigma^{AB}$.

In [5] we have shown why AdS_4 gravity with a topological term; i. e. an Einstein-Hilbert action with a cosmological constant plus Gauss-Bonnet terms can be obtained from the vacuum state of a **BF**-Chern-Simons-Higgs theory *without* introducing by *hand* the zero torsion condition imposed in the McDowell-Mansouri-Chamsedine-West construction. One of the most salient features of [5] was that a *geometric mean* relationship was found among the cosmological constant Λ_c, the Planck area λ^2 and the AdS_4 throat size squared R^2 given by $(\Lambda_c)^{-1} = (\lambda)^2(R^2)$. Upon setting the throat size to be of the order of the Hubble scale R_H and $\lambda = L_P$ (Planck scale), one recovers the observed value of the cosmological constant $L_P^{-2} R_H^{-2} = L_P^{-4}(L_P/R_H)^2 \sim 10^{-120} M_P^4$. A similar geometric mean relation is also obeyed by the condition $\lambda R = L^2 (= n\lambda^2)$ in the double scaling limit of Yang's algebra which suggests to identify the cosmological constant as $\Lambda_c = L^{-4}$. This geometric mean condition remains to be investigated further. In particular, we presented the preliminary steps how to construct a Noncommutative Gravity via the Vasiliev-Moyal star products deformations of the $SO(4,2)$ algebra used in the study of higher conformal massless spin theories in AdS spaces by taking the inverse-throat size $1/R$ as a deformation parameter of the $SO(4,2)$ algebra. A Moyal deformation of ordinary Gravity via $SU(\infty)$ gauge theories was advanced in [7].

3 Noncommutative QM in Yang's spacetime from ordinary QM in higher dimensions

In order to write wave equations in non-commuting spacetimes we start with a Hamiltonian written in *dimensionless* variables involving the terms of the relativistic oscillator (let us say oscillations of the center of mass) and the rigid rotor/top terms (rotations about the center of mass)

$$H = \left(\frac{p_\mu}{\hbar/R}\right)^2 + \left(\frac{x_\mu}{L_P}\right)^2 + \left(\Sigma^{\mu\nu}\right)^2 \tag{3.1}$$

with the fundamental difference that the coordinates x^μ and momenta p^μ obey the non-commutative Yang's space time algebra. For this reason one *cannot* naively replace p^μ any longer by the differential operator $-i\hbar\partial/\partial x^\mu$ nor write the $\Sigma^{\mu\nu}$ generators as $\frac{1}{\hbar}(x^\mu\partial_{x_\nu} - x^\nu\partial_{x_\mu})$. The correct coordinate realization of Yang's noncommutative spacetime algebra requires, for example, embedding the 4-dim space into 6-dim and expressing the coordinates and momenta operators as follows

$$\begin{aligned} \frac{p_\mu}{(\hbar/R)} &\leftrightarrow \Sigma^{\mu 6} = i\,\frac{1}{\hbar}\left(X^\mu\partial_{X_6} - X^6\partial_{X_\mu}\right), \\ \frac{x_\mu}{L_P} &\leftrightarrow \Sigma^{\mu 5} = i\,\frac{1}{\hbar}\left(X^\mu\partial_{X_5} - X^5\partial_{X_\mu}\right), \\ \Sigma^{\mu\nu} &\leftrightarrow i\,\frac{1}{\hbar}\left(X^\mu\partial_{X_\nu} - X^\nu\partial_{X_\mu}\right), \\ \mathcal{N} = \Sigma^{56} &\leftrightarrow i\,\frac{1}{\hbar}\left(X^5\partial_{X_6} - X^6\partial_{X_5}\right). \end{aligned} \tag{3.2}$$

This allows to express H in terms of the standard angular momentum operators in 6-dim. The $X^A = X^\mu, X^5, X^6$ coordinates ($\mu = 1,2,3,4$) and $P^A = P^\mu, P^5, P^6$ momentum variables obey the standard commutation relations of ordinary QM in 6-dim, namely — $\left[X^A, X^B\right] = 0$, $\left[P^A, P^B\right] = 0$, $\left[X^A, P^B\right] = i\hbar\eta^{AB}$, so that the momentum admits the standard realization as $P^A = -i\hbar\partial/\partial X_A$.

Therefore, concluding, the Hamiltonian H in eq-(3.1) associated with the non-commuting coordinates x^μ and momenta p^μ in $d-1$-dimensions can be written in terms of the standard angular momentum operators in $(d-1)+2 = d+1$-dim as $H = \mathcal{C}_2 - \mathcal{N}^2$, where $\mathcal{C}_2$ agrees precisely with the quadratic Casimir operator of the $SO(d-1,2)$ algebra in the spin $s = 0$ case,

$$\mathcal{C}_2 = \Sigma_{AB}\Sigma^{AB} = (X_A\partial_B - X_B\partial_A)(X^A\partial^B - X^B\partial^A). \tag{3.4}$$

One remarkable feature is that $\mathcal{C}_2$ also agrees with the d'Alambertian operator for the Anti de Sitter Space AdS_d of *unit radius* (throat size) $(D_\mu D^\mu)_{AdS_d}$ as shown by [18].

The proof requires to show that the d'Alambertian operator for the $d+1$-dim embedding space (expressed in terms of the X^A coordinates) is related to the d'Alambertian operator in AdS_d space of *unit* radius expressed in terms of the $z^1, z^2, \ldots, z^d$ *bulk intrinsic* coordinates as

$$(D_\mu D^\mu)_{R^{d+1}} = -\frac{\partial^2}{\partial\rho^2} - \frac{d}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}(D_\mu D^\mu)_{AdS} \Rightarrow$$
$$\mathcal{C}_2 = \rho^2 (D_\mu D^\mu)_{R^{d+1}} + \left[(d-1)+\rho\frac{\partial}{\partial\rho}\right]\rho\frac{\partial}{\partial\rho} = (D_\mu D^\mu)_{AdS_d}. \tag{3.5}$$

This result is just the hyperbolic-space generalization of the standard decomposition of the Laplace operator in spherical coordinates in terms of the radial derivatives plus a term containing the square of the orbital angular momentum operator L^2/r^2. In the case of nontrivial spin, the Casimir $C_2 = \Sigma_{AB}\Sigma^{AB} + S_{AB}S^{AB}$ has additional terms stemming from the spin operator.

The quantity $\Phi(z^1, z^2, \ldots, z^d)|_{\text{boundary}}$ restricted to the $d-1$-dim projective boundary of the conformally compactified AdS_d space (of unit throat size, whose topology is $S^{d-2}\times S^1$) is the sought-after solution to the Casimir invariant wave equation associated with the non-commutative x^μ coordinates and momenta p^μ of the Yang's algebra ($\mu = 1, 2, \ldots, d-1$). Pertaining to the boundary of the conformally compactified AdS_d space, there are two radii R_1, R_2 associated with S^{d-2} and S^1, respectively, and which must not be confused with the two scales R, L_P appearing in eq-(3.1). One can choose the units such that the present value of the Hubble scale (taking the Hubble scale as the infrared cutoff) is $R=1$. In these units the Planck scale L_P will be of the order of $L_P \sim 10^{-60}$. In essence, there has been a trade-off of two scales L_P, R with the two radii R_1, R_2.

Once can parametrize the coordinates of $AdS_d = AdS_{p+2}$ by writing there, according to [17], $X_0 = R\cosh(\rho)\cos(\tau)$, $X_{p+1} = R\cosh(\rho)\sin(\tau)$, $X_i = R\sinh(\rho)\,\Omega_i$.

The metric of $AdS_d = AdS_{p+2}$ space in these coordinates is $ds^2 = R^2\left[-(\cosh^2\rho)d\tau^2 + d\rho^2 + (\sinh^2\rho)d\Omega^2\right]$, where $0 \leqslant \rho$ and $0 \leqslant \tau < 2\pi$ are the global coordinates. The topology of this hyperboloid is $S^1\times R^{p+1}$. To study the causal structure of AdS it is convenient to unwrap the circle S^1 (closed-timelike coordinate τ) to obtain the universal covering of the hyperboloid without closed-timelike curves and take $-\infty \leqslant \tau \leqslant +\infty$. Upon introducing the new coordinate $0 \leqslant \theta < \frac{\pi}{2}$ related to ρ by $\tan(\theta) = \sinh(\rho)$, the metric is

$$ds^2 = \frac{R^2}{\cos^2\theta}\left[-d\tau^2 + d\theta^2 + (\sinh^2\rho)\,d\Omega^2\right]. \tag{3.6}$$

It is a conformally-rescaled version of the metric of the Einstein static universe. Namely, $AdS_d = AdS_{p+2}$ can be conformally mapped into one-half of the Einstein static universe, since the coordinate θ takes values $0 \leqslant \theta < \frac{\pi}{2}$ rather than $0 \leqslant \theta < \pi$. The boundary of the conformally compactified AdS_{p+2} space has the topology of $S^p \times S^1$ (identical to the conformal compactification of the $p+1$-dim Minkowski space). Therefore, the equator at $\theta = \frac{\pi}{2}$ is a *boundary* of the space with the topology of S^p. Ω_p is the solid angle coordinates corresponding to S^p and τ is the coordinate which parametrizes S^1. For a detailed discussion of AdS spaces and the AdS/CFT duality see [17].

The d'Alambertian in AdS_d space (of radius R, later we shall set $R=1$) is

$$D_\mu D^\mu = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu) = = \frac{\cos^2\theta}{R^2}\left[-\partial_\tau^2 + \frac{1}{(R\tan\theta)^p}\partial_\theta\left((R\tan\theta)^p\partial_\theta\right)\right] + \frac{\mathcal{L}^2}{R^2\tan^2\theta} \tag{3.7}$$

where $\mathcal{L}^2$ is the Laplacian operator in the p-dim sphere S^p whose eigenvalues are $l\,(l+p-1)$.

The scalar field can be decomposed as follows $\Phi = e^{\omega R\tau}Y_l(\Omega_p)G(\theta)$; the wave equation $(D_\mu D^\mu - m^2)\Phi = 0$ leads to the equation $\left[\cos^2\theta\left(\omega^2 + \partial_\theta^2 + \frac{p}{\tan\theta\cos^2\theta}\,\partial_\theta\right) + \frac{l(l+p-1)}{\tan^2\theta} - m^2R^2\right]G(\theta) = 0$, whose solution is

$$G(\theta) = (\sin\theta)^l\,(\cos\theta)^{\lambda_\pm}\;{}_2F_1(a,b,c;\sin\theta)\,. \tag{3.8}$$

The hypergeometric function is defined as

$${}_2F_1(a,b,c,z) = \sum\frac{(a)_k(b)_k}{(c)_k k!}\,z^n, \tag{3.9}$$

where $|z|<1$, $(\lambda)_0 = 1$, $(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+k-1)$, $k = 1, 2, \ldots$, while $a = \frac{1}{2}(l+\lambda_\pm-\omega R)$, $b = \frac{1}{2}(l+\lambda_\pm+\omega R)$, $c = l+\frac{1}{2}(p+1) > 0$, $\lambda_\pm = \frac{1}{2}(p+1) \pm \frac{1}{2}\sqrt{(p+1)^2+4(mR)^2}$.

The analytical continuation of the hypergeometric function for $|z| \geqslant 1$ is

$${}_2F_1(a,b,c,z) = = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \tag{3.10}$$

with *Real*$(c)>0$ and *Real*$(b)>0$. The boundary value when $\theta = \frac{\pi}{2}$ gives

$$\lim_{z\to 1^-} F(a,b,c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,. \tag{3.11}$$

Let us study the behaviour of the solution $G(\theta)$ in the massless case: $m=0$, $\lambda_- = 0$, $\lambda_+ = p+1$.

Solutions with $\lambda_+ = p+1$ yield a trivial value of $G(\theta) = 0$ at the boundary $\theta = \frac{\pi}{2}$ since $\cos\left(\frac{\pi}{2}\right)^{p+1} = 0$. Solutions with $\lambda_- = 0$ lead to $\cos(\theta)^{\lambda_-} = \cos(\theta)^0 = 1$ *prior* to taking the limit $\theta = \frac{\pi}{2}$. The expression $\cos\left(\frac{\pi}{2}\right)^{\lambda_-} = 0^0$ is ill defined. Upon using l'Hospital rule it yields 0. Thus, the limit $\theta = \frac{\pi}{2}$ must be taken afterwards the limit $\lambda_- = 0$:

$$\lim_{\theta\to\frac{\pi}{2}}\left[\cos(\theta)^{\lambda_-}\right] = \lim_{\theta\to\frac{\pi}{2}}\left[\cos(\theta)^0\right] = \lim_{\theta\to\frac{\pi}{2}}[1] = 1. \tag{3.12}$$

In this fashion the value of $G(\theta)$ is well defined and nonzero at the boundary when $\lambda_- = 0$ and leads to the value of the wavefunction at the boundary of the conformally compactified AdS_d (for $d = p + 2$ with radius R)

$$\Phi_{\text{bound}} = e^{i\omega\tau} Y_l(\Omega_p) \frac{\Gamma\left(l + \frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\omega R + \frac{l+p+1}{2}\right)\Gamma\left(-\omega R + \frac{l+p+1}{2}\right)} \,. \tag{3.13a}$$

upon setting the radius of AdS_d space to *unity* it gives

$$\Phi_{\text{bound}} = e^{i\omega\tau} Y_l(\Omega_p) \frac{\Gamma\left(l + \frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\omega + \frac{l+p+1}{2}\right)\Gamma\left(-\omega + \frac{l+p+1}{2}\right)} \,. \tag{3.13b}$$

Hence, Φ_{bound} in eq-(3.13b) is the solution to the Casimir invariant wave equation in the *massless* $m = 0$ case

$$\mathcal{C}_2\Phi = \left[\left(\frac{p_\mu}{\hbar/R}\right)^2 + \left(\frac{x_\mu}{L_P}\right)^2 + \left(\Sigma^{\mu\nu}\right)^2 + \mathcal{N}^2\right]\Phi = 0 \tag{3.14}$$

and (when $R = 1$)

$$\left[\left(\frac{p_\mu}{\hbar/R}\right)^2 + \left(\frac{x_\mu}{L_P}\right)^2 + \left(\Sigma^{\mu\nu}\right)^2\right]\Phi = \left[\mathcal{C}_2 - \mathcal{N}^2\right]\Phi = -\omega^2\Phi \tag{3.15}$$

since $\mathcal{N} = \Sigma^{56}$ is the rotation generator along the S^1 component of AdS space. It acts as $\partial/\partial\tau$ only on the $e^{i\omega R\tau}$ piece of Φ. Concluding: $\Phi(z^1, z^2, \ldots, z^d)|_{\text{boundary}}$, restricted to the $d-1$-dim projective boundary of the conformally compactified AdS_d space (of *unit* radius and topology $S^{d-2} \times S^1$) given by eq-(3.12), is the sought-after solution to the wave equations (3.13, 3.14) associated with the non-commutative x^μ coordinates and momenta p^μ of the Yang's algebra and where the indices μ range over the dimensions of the *boundary* $\mu = 1, 2, \ldots, d-1$. This suggests that QM over Yang's Noncommutative Spacetimes could be well defined in terms of ordinary QM in *higher* dimensions! This idea deserves further investigations. For example, it was argued by [16] that the *quantized* Nonabelian gauge theory in d dimensions can be obtained as the infrared limit of the corresponding *classical* gauge theory in $d + 1$-dim.

4 Star products and noncommutative QM

The ordinary Moyal star-product of two functions in phase space $f(x,p)$, $g(x,p)$ is

$$(f * g)(x,p) = \sum_s \frac{\hbar^s}{s!} \sum_{t=0}^{s} (-1)^t C(s,t) \times \\ \times \left(\partial_x^{s-t}\partial_p^t f(x,p)\right)\left(\partial_x^t \partial_p^{s-t} g(x,p)\right) \tag{4.1}$$

where $C(s,t)$ is the binomial coefficient $s!/t!(s-t)!$. In the $\hbar \to 0$ limit the star product $f * g$ reduces to the ordinary pointwise product fg of functions. The Moyal product of two functions of the $2n$-dim phase space coordinates (q_i, p_i) with $i = 1, 2 \ldots n$ is

$$(f * g)(x,p) = \sum_i \sum_s \frac{\hbar^s}{s!} \sum_{t=0}^{s} (-1)^t C(s,t) \times \\ \times \left(\partial_{x_i}^{s-t}\partial_{p_i}^t f(x,p)\right)\left(\partial_{x_i}^t \partial_{p_i}^{s-t} g(x,p)\right). \tag{4.2}$$

The noncommutative, associative Moyal bracket is

$$\{f, g\}_{\text{MB}} = \frac{1}{i\hbar}\left(f * g - g * f\right). \tag{4.3}$$

The task now is to construct *novel* Moyal-Yang star products based on the noncommutative spacetime Yang's algebra. A novel star product deformations of (super) p-brane actions based on the noncommutative spacetime Yang's algebra where the deformation parameter is $\hbar_{\text{eff}} = \hbar L_P/R$, for nonzero values of $\hbar$, was obtained in [15] The modified (noncommutative) Poisson bracket is now given by

$$\{\mathcal{F}(q^m, p^m), \mathcal{G}(q^m, p^m)\}_\Omega = \\ = (\partial_{q^m}\mathcal{F})\{q^m, q^n\}(\partial_{q^n}\mathcal{G}) + (\partial_{p^m}\mathcal{F})\{p^m, p^n\}(\partial_{p^n}\mathcal{G}) + \\ + (\partial_{q^m}\mathcal{F})\{q^m, p^n\}(\partial_{p^n}\mathcal{G}) + (\partial_{p^m}\mathcal{F})\{p^m, q^n\}(\partial_{q^n}\mathcal{G}), \tag{4.4}$$

where the entries $\{q^m, q^n\} \neq 0$, $\{p^m, p^n\} \neq 0$, and also $\{p^m, q^n\} = -\{q^n, p^m\}$ can be read from the commutators described in section 2 by simply defining the deformation parameter $\hbar_{\text{eff}} \equiv \hbar(L_P/R)$. One can generalize Yang's original 4-dim algebra to noncommutative $2n$-dim world-volumes and/or spacetimes by working with the $2n+2$-dim angular-momentum algebra $SO(d,2) = SO(p+1,2) = SO(2n,2)$.

The Noncommutative Poisson brackets $\Omega(q^m, q^n) = \{q^m, q^n\}_{\text{NCPB}}$, $\Omega(p^m, p^n) = \{p^m, p^n\}_{\text{NCPB}}$, $\Omega(q^m, p^n) = -\Omega(p^n, q^m) = \{q^m, p^n\}_{\text{NCPB}}$

$$\Omega(q^m, q^n) = \lim_{\hbar_{\text{eff}} \to 0} \frac{1}{i\,\hbar_{\text{eff}}}\left[q^m, q^n\right] = -\frac{L^2}{\hbar}\,\Sigma^{mn}, \tag{4.5a}$$

$$\Omega(p^m, p^n) = \lim_{\hbar_{\text{eff}} \to 0} \frac{1}{i\,\hbar_{\text{eff}}}\left[p^m, p^n\right] = -\frac{\hbar}{L^2}\,\Sigma^{mn}, \tag{4.5b}$$

$$\Omega(q^m, p^n) = \lim_{\hbar_{\text{eff}} \to 0} \frac{1}{i\,\hbar_{\text{eff}}}\left[q^m, p^n\right] = -\eta^{mn}, \tag{4.5c}$$

defined by above expressions, where Σ^{mn} is the "classical" $\hbar_{\text{eff}} = (\hbar L_P/R) \to 0$ limit ($R \to \infty$, $L_P \to 0$, $RL_P = L^2$, $\hbar \neq 0$) of the quantity $\Sigma^{mn} = \frac{1}{\hbar}(X^m P^n - X^n P^m)$, after embedding the $d-1$-dimensional spacetime (boundary of AdS_d) into an ordinary $(d-1)+2$-dimensional one. In the $R \to \infty$, ... limit, the AdS_d space (the hyperboloid) degenerates into a *flat* Minkowski spacetime and the coordinates q^m, p^n, in that infrared limit, *coincide* with the coordinates X^m, P^n. Concluding, in the "classical" limit ($R \to \infty, \ldots$, flat limit) one has

$$\Sigma^{mn} \equiv \frac{1}{\hbar}\left(X^m P^n - X^n P^m\right) \to \frac{1}{\hbar}\left(q^m p^n - q^n p^m\right) \tag{4.5d}$$

and then one recovers in that limit the ordinary definition of the angular momentum in terms of commuting coordinates q's and commuting momenta p's.

Denoting the coordinates (q^m, p^m) by Z^m and when the Poisson structure Ω^{mn} is given in terms of *constant* numerical coefficients, the Moyal star product is defined in terms of the deformation parameter $\hbar_{\text{eff}} = \hbar L_P/R$ as

$$(\mathcal{F} * \mathcal{G})(z) \equiv \equiv \exp\left[(i\hbar_{\text{eff}})\Omega^{mn}\partial_m^{(z_1)}\partial_n^{(z_2)}\right]\mathcal{F}(z_1)\mathcal{G}(z_2)|_{z_1=z_2=z} \tag{4.6}$$

where the derivatives $\partial_m^{(z_1)}$ act only on the $\mathcal{F}(z_1)$ term and $\partial_n^{(z_2)}$ act only on the $\mathcal{G}(z_2)$ term. In our case the generalized Poisson structure Ω^{mn} is given in terms of *variable* coefficients, it is a function of the coordinates, then $\partial\Omega^{mn} \neq 0$, since the Yang's algebra is basically an angular momentum algebra, therefore the suitable Moyal-Yang star product given by Kontsevich [11] will contain the appropriate *corrections* $\partial\Omega^{mn}$ to the ordinary Moyal star product.

Denoting by $\partial_m = \partial/\partial z^m = (\partial/\partial q^m; \partial/\partial p^m)$ the Moyal-Yang-Kontsevich star product, let us say, of the Hamiltonian $H(q,p)$ with the density distribution in phase space $\rho(q,p)$ (not necessarily positive definite), $H(q,p) * \rho(q,p)$ is

$$\begin{aligned} &H\rho + i\hbar_{\text{eff}}\Omega^{mn}(\partial_m H\partial_n\rho) + \\ &+ \frac{(i\hbar_{\text{eff}})^2}{2}\Omega^{m_1n_1}\Omega^{m_2n_2}(\partial^2_{m_1m_2}H)(\partial^2_{n_1n_2}\rho) + \\ &+ \frac{(i\hbar_{\text{eff}})^2}{3}\left[\Omega^{m_1n_1}(\partial_{n_1}\Omega^{m_2n_2}) \times\right. \\ &\times \left.(\partial_{m_1}\partial_{n_2}H\partial_{n_2}\rho - \partial_{m_2}H\partial_{m_1}\partial_{n_2}\rho)\right] + \mathrm{O}(\hbar^3_{\text{eff}}), \end{aligned} \tag{4.7}$$

where the explicit components of Ω^{mn} are given by eqs-(4.5a–4.5d). The Kontsevich star product is associative up to second order [11] $(f*g)*h = f*(g*h) + \mathrm{O}(\hbar^3_{\text{eff}})$.

The most general expression of the Kontsevich star product in Poisson manifold is quite elaborate and shall not be given here. Star products in *curved* phase spaces have been constructed by Fedosov [12]. Despite these technical subtleties it did not affect the final expressions for the "classical" Noncommutative p-brane actions as shown in [15] when one takes the $\hbar_{\text{eff}} \to 0$ "classical" limit. In that limit there are still *nontrivial noncommutative corrections* to the ordinary p-brane actions.

In the Weyl-Wigner-Gronewold-Moyal quantization scheme in phase spaces one writes

$$H(x,p) * \rho(x,p) = \rho(x,p) * H(x,p) = E\rho(x,p), \tag{4.8}$$

where the Wigner density function in phase space associated with the Hilbert space state $|\Psi>$ is

$$\rho(x,p,\hbar) = \frac{1}{2\pi}\int dy\, \Psi^*\left(x - \frac{\hbar y}{2}\right)\Psi\left(x + \frac{\hbar y}{2}\right)e^{\frac{ipy}{\hbar}} \tag{4.9}$$

plus their higher dimensional generalizations. It remains to be studied if this Weyl-Wigner-Gronewold-Moyal quantization scheme is appropriate to study QM over Noncommutative Yang's spacetimes when we use the above Moyal-Yang-Kontsevich star products. A recent study of the Yang's Noncommutative algebra and *discrete* Hilbert (Buniy-Hsu-Zee) spaces was undertaken by Tanaka [3].

Let us write down the Moyal-Yang-Konstevich star deformations of the field theory Lagrangian corresponding to the scalar field $\Phi = \Phi(X^{AB})$ which depends on the holographic-area coordinates X^{AB} [13]. The reason one should *not* try to construct the star product of $\Phi(x^m) * \Phi(x^n)$ based on the Moyal-Yang-Kontsevich product, is because the latter star product given by eq-(4.7) will introduce explicit *momentum* terms in the r.h.s of $\Phi(x^m) * \Phi(x^m)$, stemming from the expression $\Sigma^{mn} = x^m p^n - x^n p^m$ of eq-(4.5d), and thus it invalidates writing $\phi = \phi(x)$ in the first place. If the Σ^{mn} were *numerical constants*, like Θ^{mn}, then one could write the $\Phi(x^m) * \Phi(x^m)$ in a straightforward fashion as it is done in the literature.

The reason behind choosing $\Phi = \Phi(X^{AB})$ is more clear after one invokes the area-coordinates and angular momentum correspondence discussed in detail in section 2. It allows to properly define the star products. A typical Lagrangian is

$$\begin{aligned} \mathcal{L} = &-\Phi * \partial^2_{X^{AB}}\Phi\left(X^{AB}\right) + \frac{m^2}{2}\Phi\left(X^{AB}\right) * \Phi\left(X^{AB}\right) + \\ &+ \frac{g^n}{n}\Phi\left(X^{AB}\right) * \Phi\left(X^{AB}\right) * \cdots *_n \Phi\left(X^{AB}\right) \end{aligned} \tag{4.10}$$

and leads to the equations of motion

$$\begin{aligned} &-\left(\partial/\partial X^{AB}\right)\left(\partial/\partial X^{AB}\right)\Phi\left(X^{AB}\right) + m^2\Phi\left(X^{AB}\right) + \\ &+ g^n\,\Phi\left(X^{AB}\right) * \Phi\left(X^{AB}\right) * \cdots *_{n-1} \Phi\left(X^{AB}\right) = 0 \end{aligned} \tag{4.11}$$

when the multi-symplectic Ω^{ABCD} form is coordinate-independent, the star product is

$$\begin{aligned} &(\Phi * \Phi)(Z^{AB}) \equiv \exp\left[\left(i\lambda\Omega^{ABCD}\partial_{X^{AB}}\partial_{Y^{AB}}\right)\right] \times \\ &\times \Phi(X^{AB})\Phi\left(Y^{AB}\right)\Big|_{X=Y=Z} = \\ &= \exp\left[\left(\Sigma^{ABCD}\partial_{X^{AB}}\partial_{Y^{AB}}\right)\right]\Phi\left(X^{AB}\right)\Phi\left(Y^{AB}\right)\Big|_{X=Y=Z} \end{aligned} \tag{4.12}$$

where Σ^{ABCD} is derived from the structure constants of the holographic area-coordinate algebra in C-spaces [14] as: $\left[X^{AB}, X^{CD}\right] = \Sigma^{ABCD} \equiv iL_P^2\left(\eta^{AD}X^{BC} - \eta^{AC}X^{BD} + \eta^{BC}X^{AD} - \eta^{BD}X^{AC}\right)$. There are nontrivial derivative terms acting on Σ^{ABCD} in the definition of the star product $(\Phi * \Phi)(Z^{MN})$ as we have seen in the definition of the Kontsevich star product $H(x,p) * \rho(x,p)$ in eq-(4.7). The expansion parameter in the star product is the Planck scale squared $\lambda = L_P^2$. The star product has the same functional form as (4-7) with the only difference that now we are taking derivatives w.r.t the area-coordinates X^{AB} instead of derivatives w.r.t the variables x, p, hence to order $\mathrm{O}(L_P^4)$, the star product is

$$\begin{aligned} \Phi * \Phi = \;&\Phi^2 + \Sigma^{ABCD}(\partial_{AB}\Phi\,\partial_{CD}\Phi) + \\ &+ \frac{1}{2}\Sigma^{A_1B_1C_1D_1}\Sigma^{A_2B_2C_2D_2}(\partial^2_{A_1B_1A_2B_2}\Phi)(\partial^2_{C_1D_1C_2D_2}\Phi) + \\ &+ \frac{1}{3}\left[\Sigma^{A_1B_1C_1D_1}(\partial_{C_1D_1}\Sigma^{A_2B_2C_2D_2}) \times\right. \\ &\times \left.(\partial_{A_1B_1}\partial_{A_2B_2}\Phi\,\partial_{C_2D_2}\Phi - B_1 \leftrightarrow B_2)\right]. \end{aligned} \tag{4.13}$$

Notice that the powers of iL_P^2 are encoded in the definition of Σ^{ABCD}. The star product is noncommutative but is also nonassociative at the order $O(L_P^6)$ and beyond. The Jacobi identities would be anomalous at that order and beyond. The derivatives acting on Σ^{ABCD} are

$$(\partial_{C_1 D_1}\Sigma^{A_2 B_2 C_2 D_2}) = \\ = i L_P^2 \left(\eta^{A_2 D_2}\,\delta^{B_2 C_2}_{C_1 D_1} - \eta^{A_2 C_2}\,\delta^{B_2 D_2}_{C_1 D_1}\right) + \\ + i L_P^2 \left(\eta^{B_2 C_2}\,\delta^{A_2 D_2}_{C_1 D_1} - \eta^{B_2 D_2}\,\delta^{A_2 C_2}_{C_1 D_1}\right). \tag{4.14}$$

where $\delta^{AB}_{CD} = \delta^A_C \delta^B_D - \delta^A_D \delta^B_C$ and the higher derivatives like $\partial^2_{A_1 B_1 C_1 D_1}\Sigma^{A_2 B_2 C_2 D_2}$ will be zero.

5 On the generalized Dirac-Konstant equation in Clifford spaces

To conclude this work we will discuss the wave equations relevant to fermions. The "square" of the Dirac-Konstant equation $(\gamma^{[\mu\nu]}\Sigma_{\mu\nu})\Psi = \lambda\Psi$ yields

$$(\gamma^{[\mu\nu]}\gamma^{[\rho\tau]}\Sigma_{\mu\nu}\Sigma_{\rho\tau})\Psi = \lambda^2\Psi \;\Rightarrow \\ \Big[\gamma^{[\mu\nu\rho\tau]} + (\eta^{\mu\rho}\gamma^{[\nu\tau]} - \eta^{\mu\tau}\gamma^{[\nu\rho]} + \dots) + \\ + (\eta^{\mu\rho}\eta^{\nu\tau}\mathbf{1} - \eta^{\mu\tau}\eta^{\nu\rho}\mathbf{1})\Big]\,\Sigma_{\mu\nu}\Sigma_{\rho\tau}\Psi = \lambda^2\Psi \tag{5.2}$$

where we omitted numerical factors. The generalized Dirac equation in Clifford spaces is given by [13]

$$-i\left(\frac{\partial}{\partial\sigma} + \gamma^{\mu}\frac{\partial}{\partial x^{\mu}} + \gamma^{[\mu\nu]}\frac{\partial}{\partial x^{\mu\nu}} + \dots \right.\\ \left. + \gamma^{[\mu_1\mu_2\dots\mu_d]}\frac{\partial}{\partial x^{\mu_1\mu_2\dots\mu_d}}\right)\Psi = \lambda\Psi\,, \tag{5.3}$$

where $\sigma, x^{\mu}, x^{\mu\nu}, \dots$ are the generalized coordinates associated with the Clifford *polyvector* in C-space

$$X = \sigma\mathbf{1} + \gamma^{\mu}x_{\mu} + \gamma^{\mu_1\mu_2}x_{\mu_1\mu_2} + \dots\gamma^{\mu_1\mu_2\dots\mu_d}x_{\mu_1\mu_2\dots\mu_d} \tag{5.4}$$

after the length scale expansion parameter is set to unity. The generalized Dirac-Konstant equations in Clifford-spaces are obtained after introducing the generalized angular momentum operators [14]

$$\Sigma^{[[\mu_1\mu_2\dots\mu_n][\nu_1\nu_2\dots\nu_n]]} = X^{[[\mu_1\mu_2\dots\mu_n]}P^{[\nu_1\nu_2\dots\nu_n]]} = \\ = X^{[\mu_1\mu_2\dots\mu_n]}\frac{i\partial}{\partial X_{[\nu_1\nu_2\dots\nu_n]}} - X^{[\nu_1\nu_2\dots\nu_n]}\frac{i\partial}{\partial X_{[\mu_1\mu_2\dots\mu_n]}} \tag{5.5}$$

by writing

$$\sum_n \gamma^{[[\mu_1\mu_2\dots\mu_n]}\gamma^{[\nu_1\nu_2\dots\nu_n]]}\Sigma_{[[\mu_1\mu_2\dots\mu_n][\nu_1\nu_2\dots\nu_n]]}\Psi = \lambda\Psi \tag{5.6}$$

and where we sum over all polyvector-valued indices (antisymmetric tensors of arbitrary rank). Upon squaring eq-(5.4), one obtains the Clifford space extensions of the $D0$-brane field equations found in [3] which are of the form

$$\left[X^{AB}\frac{\partial}{\partial X_{CD}} - X^{CD}\frac{\partial}{\partial X_{AB}}\right]\times \\ \times\left[X_{AB}\frac{\partial}{\partial X^{CD}} - X_{CD}\frac{\partial}{\partial X^{AB}}\right]\Psi = 0\,, \tag{5.6}$$

where $A, B = 1, 2, \dots, 6$. It is warranted to study all these equations in future work and their relation to the physics of D-branes and Matrix Models [3]. Yang's Noncommutative algebra should be extended to superspaces, meaning non-anti-commuting Grassmanian coordinates and noncommuting bosonic coordinates.

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References

1. Yang C. N. *Phys. Rev.*, 1947, v. 72, 874; *Proc. of the Intern. Conference on Elementary Particles*, 1965, Kyoto, 322–323.
2. Snyder S. *Phys. Rev.*, 1947, v. 71, 38; *ibid.*, 1947, v. 71, 78.
3. Tanaka S. *Nuovo Cimento*, 1999, v. 114B, 49; arXiv: hep-th/0511023; hep-th/0406166; hep-th/0002001; hep-th/0303105.
4. Ashtekar A., Rovelli C., and Smolin L. *Phys. Rev. Lett.*, 1992, v. 69, 237.
5. Castro C. *Mod. Phys. Lett.*, 2002, v. A17, No. 32, 2095.
6. Castro C. arXiv: hep-th/9908115; Ansoldi S., Castro C., and Spallucci E. *Phys. Lett.*, 2001, v. B504, 174; *Class. Quan. Gravity*, 2001, v. 18, L17–L23; *Class. Quan. Gravity*, 2001, v. 18, 2865; Ansoldi S., Castro C., Guendelman E., and Spallucci E. *Class. Quant. Gravity*, 2002, v. 19, L135.
7. Castro C. *General Relativity and Gravitation*, 2004, v. 36, No. 12, 2605.
8. Castro C. *Jour. Math. Phys.*, 1993, v. 34, 681; *Phys. Lett.*, 1992, v. B288, 291.
9. Park Q. H. *Int. Jour. Mod. Phys.*, 1992, v. A7, 1415; Garcia-Compean H., Plebanski J., and Przanowski M. arXiv: hep-th/9702046.
10. Hoppe J. Quantum theory of a relativistic surface. Ph.D Thesis, MIT, 1982.
11. Kontsevich M. arXiv: q-alg/9709040.
12. Fedosov B. *J. Diff. Geom.*, 1994, v. 40, No. 2, 213.
13. Castro C. and Pavšič M. *Progress in Physics*, 2005, v. 1, 31–64.
14. Castro C. *Foundations of Physics*, 2005, v. 35, No. 6, 971.
15. Castro C. *Phys. Lett.*, 2005, v. B626, 209; *Foundations of Physics*, 2000, v. 8, 1301.
16. Biro T., Mueller B., and Matinyan S. arXiv: hep-th/0301131.
17. Aharony O., Gubser S., Maldacena J., Ooguri H., and Oz Y. *Phys. Rep.*, 2000, v. 323, 183.
18. de Witt B. and Herger I. arXiv: hep-th/9908005.