On the Regge-Wheeler Tortoise and the Kruskal-Szekeres Coordinates

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The Regge-Wheeler tortoise "coordinate" and the the Kruskal-Szekeres "extension" are built upon a latent set of invalid assumptions. Consequently, they have led to fallacious conclusions about Einstein's gravitational field. The persistent unjustified claims made for the aforesaid alleged coordinates are not sustained by mathematical rigour. They must therefore be discarded.

1 Introduction

The Regge-Wheeler tortoise coordinate was not conjured up from thin air. On the contrary, is was obtained *a posteriori* from the Droste/Weyl/(Hilbert) [1, 2, 3] (the DW/H) metric for the static vacuum field; or, more accurately, from Hilbert's corruption of the spacetime metric obtained by Johannes Droste.

The first presentation and misguided use of the Regge-Wheeler coordinate was made by A. S. Eddington [4] in 1924. Finkelstein [5], years later, in 1958, presented much the same; since then virtually canonised in the so-called "Eddington-Finkelstein" coordinates. Kruskal [6], and Szekeres [7], in 1960, compounded the errors with additional errors, all built upon the very same fallacious assumptions, by adding even more fallacious assumptions. The result has been a rather incompetent use of mathematics to produce nonsense on an extraordinary scale.

Orthodox relativists are now so imbued with the misconceptions that they are, for the most part, no longer capable of rational thought on the subject. Although the erroneous assumptions of the orthodox have been previously demonstrated to be false [8–18] they have consistently and conveniently ignored the proofs.

I amplify the erroneous assumptions of the orthodox relativists in terms of the Regge-Wheeler tortoise, and consequently in the Kruskal-Szekeres phantasmagoria.

2 The orthodox confusion and delusion

Consider the DW/H line-element

$$ds^2 = \left(1 - \frac{\alpha}{r}\right) dt^2 - \left(1 - \frac{\alpha}{r}\right)^{-1} dr^2 - \\ - r^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \tag{1}$$

where $\alpha = 2m$. Droste showed that $\alpha < r < \infty$ is the correct domain of definition on (1), as did Weyl some time later. Hilbert however, claimed $0 < r < \infty$. Modern orthodox relativists claim two intervals, $0 < r < \alpha$, $\alpha < r < \infty$, and call the latter the "exterior" Schwarzschild solution and the former a "black hole", notwithstanding that (1) with $0 < r < \infty$ was never proposed by K. Schwarzschild [19]. Astonishingly, the vast majority of orthodox relativists, it seems, have never even heard of Schwarzschild's true solution.

I have proved elsewhere [11, 12, 13] that the orthodox, when considering (1), have made three invalid assumptions, to wit

(a) r is a proper radius;

(b) r can go down to zero;

(c) A singularity must occur where the Riemann tensor scalar curvature invariant (the Kretschmann scalar), $f = R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}$, is unbounded.

None of these assumptions have ever been proved true with the required mathematical rigour by any orthodox relativist. Notwithstanding, they blindly proceed on the assumption that they are all true. The fact remains however, that they are all demonstrably false.

Consider assumption (a). By what rigorous argument have the orthodox identified r as a radial quantity on (1)? Moreover, by what rigorous mathematical means have they ever indicated what they mean by a radial quantity on (1)? Even a cursory reading of the literature testifies to the fact that the orthodox relativists have never offered any mathematical rigour to justify assumption (a). Mathematical rigour actually proves that this assumption is false.

Consider assumption (b). By what rigorous means has it ever been proved that r can go down to zero on (1)? The sad fact is that the orthodox have never offered a rigorous argument. All they have ever done is inspect (1) and claim that there are singularities at $r = \alpha$ and at $r = 0$, and thereafter concocted means to make one of them ($r = 0$) a "physical" singularity, and the other a "coordinate" singularity, and vaguely refer to the latter as a "pathology" of coordinates, whatever that means. The allegation of singularities at $r = \alpha$ and at $r = 0$ also involves the unproven assumption (a). Evidently the orthodox consider that assumptions (a) and (b) are self-evident, and so they don't even think about them. However, assumptions (a) and (b) are not self-evident and if they are to be justifiably used, they must first be proved. No orthodox relativist has ever bothered to attempt the necessary proofs. Indeed, none it would seem have ever seen the need for proofs, owing to their "self-evident" assumptions.

Assumption (c) is an even more curious one. Indeed, it is actually a bit of legerdemain. Having just assumed (a) and (b), the orthodox needed some means to identify their "physical" singularity. They went looking for it at a suitable unbounded curvature scalar, found it in the Kretschmann scalar, after a series of misguided transformations of "coordinates" leading to the Kruskal-Szekeres "extension", and thereafter have claimed singularity of the Kretschmann type in the static vacuum field.

Furthermore, using these unproved assumptions, the orthodox relativists have claimed a process of "gravitational collapse" to a "point-mass". And with this they have developed what they have called grandiosely and misguidedly, "singularity theorems", by which it is alleged that "physical" singularities and "trapped surfaces" are a necessary consequence of gravitational collapse, and even cosmologically, called Friedmann singularities.

The orthodox relativists must first prove their assumptions by rigorous mathematics. Unless they do this, their analyses are unsubstantiated and cannot be admitted.

Since the orthodox assumptions have in fact already been rigorously proved entirely false, the theory that the orthodox have built upon them is also false.

3 The Regge-Wheeler tortoise; the Kruskal-Szekeres phantasmagoria

Since the Regge-Wheeler tortoise does not come from thin air, from where does it come?

First consider the general static line-element

$$ds^2 = A\left(\sqrt{C(r)}\right)dt^2 - B\left(\sqrt{C(r)}\right)d\sqrt{C(r)}^{\,2} - C(r)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \quad (2)$$

$$A, B, C > 0\,.$$

It has the solution

$$ds^2 = \left(1 - \frac{\alpha}{\sqrt{C(r)}}\right)dt^2 - \left(1 - \frac{\alpha}{\sqrt{C(r)}}\right)^{-1} d\sqrt{C(r)}^{\,2} - C(r)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \quad (3)$$

and setting $R_c(r) = \sqrt{C(r)}$ for convenience, this becomes

$$ds^2 = \left(1 - \frac{\alpha}{R_c(r)}\right)dt^2 - \left(1 - \frac{\alpha}{R_c(r)}\right)^{-1} dR_c^2(r) - R_c^2(r)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \quad (4)$$

for some analytic function $R_c(r)$. Clearly, if $R_c(r)$ is set equal to r, then (1) is obtained.

Reduce (4) to two dimensions, thus

$$ds^2 = \left(1 - \frac{\alpha}{R_c(r)}\right)dt^2 - \left(1 - \frac{\alpha}{R_c(r)}\right)^{-1} dR_c^2(r)\,. \quad (5)$$

The null geodesics are given by

$$ds^2 = 0 = \left(1 - \frac{\alpha}{R_c(r)}\right)dt^2 - \left(1 - \frac{\alpha}{R_c(r)}\right)^{-1} dR_c^2(r)\,.$$

Consequently

$$\left(\frac{dt}{dR_c(r)}\right)^2 = \left(\frac{R_c(r)}{R_c(r) - \alpha}\right)^2,$$

and therefore,

$$t = \pm\left[R_c(r) + \alpha \ln\left|\frac{R_c(r)}{\alpha} - 1\right|\right] + \text{const.}$$

Now

$$R^*(r) = R_c(r) + \alpha \ln\left|\frac{R_c(r)}{\alpha} - 1\right| \quad (6)$$

is the so-called Regge-Wheeler tortoise coordinate. If $R_c(r) = r$, then

$$r^* = r + \alpha \ln\left|\frac{r}{\alpha} - 1\right|, \quad (7)$$

which is the standard expression used by the orthodox. They never use the general expression (6) because they only ever consider the particular case $R_c(r) = r$, owing to the fact that they do not know that their equations relate to a particular case. Furthermore, with their unproven and invalid assumptions (a) and (b), many orthodox relativists claim

$$0 = 0 + \alpha \ln\left|\frac{0}{\alpha} - 1\right| \quad (8)$$

so that $r_0^* = r_0 = 0$. But as explained above, assuming $r_0 = 0$ in (1) has no rigorous basis, so (8) is rather misguided.

Let us now consider (2). I identify therein the radius of curvature $R_c(r)$ as the square root of the coefficient of the angular terms, and the proper radius $R_p(r)$ as the integral of the square root of the component of the metric tensor containing the squared differential element of the radius of curvature. Thus, on (2),

$$R_c(r) = \sqrt{C(r)}\,, \qquad R_p(r) = \int \sqrt{B(\sqrt{C(r)})}\; d\sqrt{C(r)} \;+\; \text{const.} \quad (9)$$

In relation to (4) it follows that,

$R_c(r)$ is the radius of curvature,

$$R_p(r) = \int \sqrt{\frac{R_c(r)}{(R_c(r) - \alpha)}}\; dR_c(r) + K\,, \quad (10)$$

where K is a constant to be rigorously determined by a boundary condition. Note that according to (10) there is no *a priori* reason for $R_p(r)$ and $R_c(r)$ to be identical in Einstein's gravitational field.

Now consider the usual Minkowski metric,

$$ds^2 = dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{11}$$

$$0 \leqslant r < \infty,$$

where

$$R_c(r) = r, \qquad R_p(r) = \int_0^r dr = r \equiv R_c(r). \tag{12}$$

In this case $R_p(r)$ is identical to $R_c(r)$. The identity is due to the fact that the spatial components of Minkowski space are Efcleethean*. But (4), and hence (10), are non-Efcleethean, and so there is no reason for $R_p(r)$ and $R_c(r)$ to be identical therein.

The geometry of a spherically symmetric line-element is an intrinsic and invariant property, by which radii are rigorously determined. The radius of curvature is always the square root of the coefficient of the angular terms and the proper radius is always the integral of the square root of the component containing the square of the differential element of the radius of curvature. Note that in general $R_c(r)$ and $R_p(r)$ are analytic functions of r, so that r is merely a parameter, and not a radial quantity in (2) and (4). So $R_c(r)$ and $R_p(r)$ map the parameter r into radii (i. e. distances) in the gravitational field. Note further that r is actually defined in Minkowski space. Thus, a distance in Minkowski space is mapped into corresponding distances in Einstein's gravitational field by the mappings $R_c(r)$ and $R_p(r)$.

It has been proved [11, 12] that the admissible form for $R_c(r)$ is,

$$R_c(r) = \left(\left|r - r_0\right|^n + \alpha^n\right)^{\frac{1}{n}}, \tag{13}$$

$$n \in \Re^+, \quad r_0 \in \Re, \quad \alpha = 2m, \quad r \neq r_0,$$

where n and r_0 are entirely arbitrary constants, and that

$$R_p(r) = \sqrt{R_c(r)\left(R_c(r) - \alpha\right)} + $$

$$+ \alpha \ln\left|\frac{\sqrt{R_c(r)} + \sqrt{R_c(r) - \alpha}}{\sqrt{\alpha}}\right|.$$

If $n = 1$, $r_0 = \alpha$, $r > r_0$ are chosen, then by (13), $R_c(r) = r$ and equation (1) is recovered; but by (13), $\alpha < r < \infty$ is then the range on the r-parameter. Note that in this case

$$R_c(\alpha) = \alpha, \qquad R_p(\alpha) = 0,$$

and that in general,

$$R_c(r_0) = \alpha, \qquad R_p(r_0) = 0,$$

$$\alpha < R_c(r) < \infty,$$

since the value of r_0 is immaterial. I remark in passing that if $n = 3$, $r_0 = 0$, $r > 0$ are chosen, Schwarzschild's original solution results.

Returning now to the Regge-Wheeler tortoise, it is evident that

$$-\infty < R^*(r) < \infty,$$

and that $R^*(r) = 0$ when $R(r) \approx 1.278465\,\alpha$. Now according to (13), $\alpha < R_c(r) < \infty$, so the Regge-Wheeler tortoise can be written generally as,

$$R^*(r) = R_c(r) + \alpha \ln\left(\frac{R_c(r)}{\alpha} - 1\right), \tag{14}$$

which is, in the particular case invariably used by the orthodox relativists,

$$r^* = r + \alpha \ln\left(\frac{r}{\alpha} - 1\right),$$

and so, by (13) and (14), the orthodox claim that

$$0 = 0 + \alpha \ln\left|\frac{0}{\alpha} - 1\right|,$$

is nonsense. It is due to the invalid assumptions (a) and (b) which the orthodox relativists have erroneously taken for granted. Of course, the tortoise, r^*, cannot be interpreted as a radius of curvature, since in doing so would violate the intrinsic geometry of the metric. This is clearly evident from (13), which specifies the permissible form of a radius of curvature on a metric of the form (4).

So what is the motivation to the Regge-Wheeler tortoise and the subsequent Kruskal-Szekeres extension? Very simply this, to rid (1) of the singularity at $r = \alpha$ and make $r = 0$ a "physical" singularity, satisfying the *ad hoc* assumption (c), under the mistaken belief that† $r = \alpha$ is not a physical singularity (but it is a true singularity, however, not a Kretschmann curvature-type). This misguided notion is compounded by a failure to realise that there are two radii in Einstein's gravitational field and that they are never identical, except in the infinitely far field where spacetime becomes Minkowski, and that what they treat as a proper radius in the gravitational field is in fact the radius of curvature in their particular metric, which cannot go down to zero. Only the proper radius can approach zero, although it cannot take the value of zero, i. e. $r \neq r_0$ in (13), since $R_p(r_0) \equiv 0$ marks the location of the centre of mass of the source of the field, which is not a physical object.

*Owing to the geometry due to Efcleethees, for those ignorant of Greek; usually and incorrectly Euclid.

†Indeed, that $\left(R_c(r_0) \equiv \alpha\right) \equiv \left(R_p(r_0) \equiv 0\right)$.

The mechanical procedure to the Kruskal-Szekeres extension is well-known, so I will not reproduce it here, suffice to say that it proposes the following null coordinates u and v,

$$u = t - R^*(r)\,,$$

$$v = t + R^*(r)\,,$$

which is always given by the orthodox relativists in the particular case

$$u = t - r^*\,,$$

$$v = t + r^*\,.$$

Along the way to the Kruskal-Szekeres extension, the sole purpose of which is to misguidedly drive the radius of curvature r in (1) down to zero, owing to their invalid assumptions (a), (b) and (c), the orthodox obtain

$$ds^2 = -\,\frac{\alpha e^{-\frac{r}{\alpha}}}{r}\, e^{\frac{(v-u)}{2\alpha}}\, du\, dv\,,$$

which in general terms is

$$ds^2 = -\,\frac{\alpha e^{-\frac{R_c(r)}{\alpha}}}{R_c(r)}\, e^{\frac{(v-u)}{2\alpha}}\, du\, dv\,,$$

and erroneously claim that the metric components of (1) have been factored into a piece, $\frac{e^{-r/\alpha}}{r}$, which is non-singular as $r \to \alpha$, times a piece with u and v dependence [20]. The claim is of course completely spurious, since it is based upon the false assumptions (a), (b), and (c). The orthodox relativists have not, contrary to their claims, developed a coordinate patch to cover a part of an otherwise incompletely covered manifold. What they have actually done, quite unwittingly, is invent a completely separate manifold, which they glue onto the manifold of the true Schwarzschild field, and confound this new and separate manifold as a part of the original manifold, and by means of the Kruskal-Szekeres extension, leap between manifolds in the mistaken belief that they are moving between coordinate patches on one manifold. The whole procedure is ludicrous; and patently false. Loinger [21] has also noted that the alleged "interior" of the Hilbert solution is a different manifold.

The fact that the Hilbert solution is not diffeomorphic to Schwarzschild's solution was proved by Abrams [9], who showed that the Droste/Weyl metric is diffeomorphic to Schwarzschild's original solution. This is manifest in (13), and can be easily demonstrated alternatively by a simple transformation, as follows. In the Hilbert metric, denote the radius of curvature by r^*, and equate this to Schwarzschild's radius of curvature thus,

$$r^* = \left(r^3 + \alpha^3\right)^{\frac{1}{3}}\,. \qquad (15)$$

Since $0 < r < \infty$ in Schwarzschild's original solution, it follows from this that

$$\alpha < r^* < \infty\,,$$

which is precisely what Droste obtained; later confirmed by Weyl. There is no "interior" associated with the DW/H metric, and no "trapped surface". The transformation (15) simply shifts the location of the centre of mass of the source in parameter space from $r_0 = 0$ to $r_0 = \alpha$, as given explicitly in (13).

4 Recapitulation and general comments

The standard school of relativists has never attempted to rigorously prove its assumptions about the variable r appearing in the line-element (1). It has never provided any rigorous argument as to what constitutes a radial quantity in Einstein's gravitational field. It has invented a curvature condition, in the behaviour of the Kretschmann scalar, as an *ad hoc* basis for singularity in Einstein's gravitational field.

The Regge-Wheeler tortoise has been thoroughly misinterpreted by the standard school of relativists. The Kruskal-Szekeres extension is a misguided procedure, and does not lead to a coordinate patch, but in fact, to a completely separate manifold having nothing to do with a Schwazschild space. The motivation to the Eddington-Finkelstein coordinates and the Kruskal-Szekeres extension is due to the erroneous assumptions that the variable r in (1) is a proper radius and can therefore go down to zero.

The standard school has failed to see that there are two radii in Einstein's gravitational field, which are determined by the intrinsic geometry of the metric. Thus, it has failed to understand the geometrical structure of type 1 Einstein spaces. Consequently, the orthodox relativists have incorrectly treated the variable r in (1) as a proper radius, failing to see that it is in fact the radius of curvature in (1), and that the proper radius must in fact be calculated by the geometrical relations intrinsic to the metric. They have failed to realise that the quantity r is in general nothing more than a parameter, defined in Minkowski space, which is mapped into the radii of the gravitational field, thereby making Minkowski space a parameter space from which Efcleethean distance is mapped into the corresponding true radii of Einstein's pseudo-Riemannian gravitational field.

The so-called "singularity theorems" are not theorems at all, as they are based upon false concepts. The "point-mass" is actually nothing more than the location of the centre of mass of the source of the gravitational field, and has no physical significance. Moreover, the alleged theorems are based upon the invalid construction of "trapped surfaces", essentially derived from the false assumptions (a), (b) and (c). The Friedmann singularities simply do not exist at all, either physically or mathematically, as it has been rigorously proved that cosmological solutions for isotropic type 1 Einstein spaces do not even exist [14], so that the Standard Cosmological model is completely invalid.

My own experience has been that most orthodox relativists just ignore the facts, resort to aggressive abuse when confronted with them, and merrily continue with their demonstrably false assumptions. But here is a revelation: abuse and ignorance are not scientific methods. Evidently, scientific method is no longer required in science.

I have in the past, invited certain very substantial (and some not so substantial) elements of the orthodox relativists, literally under a torrent of vicious abuse, both gutter and eloquent, depending upon the person, (to which I have on occasion responded in kind after enduring far too much), to prove their assumptions (a), (b), and (c). Not one of them took up the invitation. I have also invited them to prove me wrong by simply providing a rigorous demonstration that the radius of curvature is not always the square root of the coefficient of the angular terms of the metric, and that the proper radius is not always the integral of the square root of the component containing the square of the differential element of the radius of curvature. Not one of them has taken up that invitation either. To refute my analysis is very simple in principle — rigorously prove the foregoing.

Alas, the orthodox are evidently unwilling to do so, being content instead to foist their errors upon all and sundry in the guise of profundity, to salve their need of vainglory, and ignore or abuse those who ask legitimate questions as to their analyses. And quite a few persons who have pointed out serious errors in the standard theory, have been refused any and all opportunity to publish papers on these matters in those journals and electronic archives which constitute the stamping grounds of the orthodox.

I give the foregoing in illustration of how modern science is now being deliberately censored and falsified. This cannot be allowed to continue, and those responsible must be exposed and penalised. It is my view that what the modern orthodox relativists have done amounts to scientific fraud. The current situation is so appalling that to remain silent would itself be criminal.

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