Fractal Scaling Models of Resonant Oscillations in Chain Systems of Harmonic Oscillators

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Logarithmic scaling invariance is a wide distributed natural phenomenon and was proved in the distributions of physical properties of various processes — in high energy physics, chemistry, seismicity, biology, geology and technology. Based on the Gantmacher-Krein continued fraction method the present paper introduces fractal scaling models of resonant oscillations in chain systems of harmonic oscillators. These models generate logarithmic scaling spectra. The introduced models are not based on any statements about the nature of the link or interaction between the elements of the oscillating system. Therefore the model statements are quite generally, what opens a wide field of possible applications.

1 Introduction

Within the past 40 years many articles were published which show that logarithmic scaling invariance ("Scaling") is a wide distributed natural phenomenon.

In 1967/68 Feynman and Bjorken [1] discovered the scaling phenomenon in high energy physics, concrete in hadron collisions.

Simon E. Shnoll [2] found scaling in the distributions of macroscopic fluctuations of nuclear decay rates. Since 1967 his team discovers fractal scaling in the fluctuation distributions of different physical and chemical processes, as well as in the distributions of macroscopic fluctuations of different noise processes.

Within the fifties Beno Gutenberg and Charles Richter [3] have shown, that exists a logarithmic invariant (scaling) relationship between the energy (magnitude) and the total number of earthquakes in any given region and time period.

In 1981, Leonid L. Čislenko [4] published his extensive work on logarithmic invariance of the distribution of biological species, dependent on body size and weight of the organisms. By introducing a logarithmic scale for biologically significant parameters, such as mean body weight and size, Čislenko was able to prove that sections of increased specie representation repeat themselves in equal logarithmic intervals.

Knut Schmidt-Nielsen [5] (1984) was able to prove scaling in biological metabolic processes.

Alexey Zhirmunsky and Viktor Kuzmin [6] (1982) discovered process-independent scaling in the development stages of embryo-, morpho- and ontogenesis and in geological history.

In 1987–1989 we [7] have shown, that fractal scaling distributions of physical process properties can be understood as a consequence of resonant oscillations of matter. Based on a fractal scaling proton resonance model, we developed methods of optimization and prognostication of technical processes, which have got european and international patents [8].

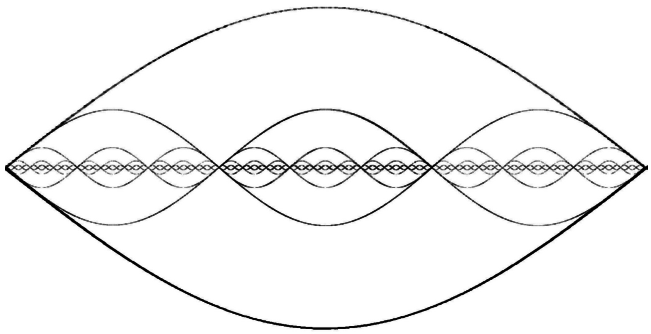

Fig. 1:

In the following we will show that Scaling is a fundamental property of any natural oscillation process. Therefore one can suspect, that natural oscillations of matter generate scaling distributions of physical properties in very different processes.

2 Fractal scaling as a fundamental property of resonant oscillations

A standing wave [9] in a homogeneous space arises only if in the direction of the wave penetration the space is finite and if the half wave length is equal to an integer part of the medium size L.

As a consequence we can find for any low enough resonant oscillation mode frequency f_0 a higher mode frequency f_1 with an integer relationship $n = f_1 / f_0$. The frequencies of such resonant oscillation modes generate exponential series:

$$f_{n,p} = f_0 \cdot n^p . \quad (1)$$

Fig. 1 illustrates the situation with $n = 3$ and $p = 0, 1, 2, \ldots$ for transversal oscillations.

Therefore, the complete resonant oscillation frequency spectrum can be represented as a set of logarithmic fractal spectra (1) with natural $n = 2, 3, 5, \ldots$ In this representation the generation of the complete resonant oscillation frequency spectrum can be understood as an arithmetical task, what can be reduced to the fundamental theorem of arithmetic, that every natural number greater than 1 can be written as a unique product of prime numbers.

In the oscillation nodes of the logarithmic fractal oscillation modes the spectral density is maximum. Where the amplitudes of the oscillation modes are maximum, the medium particles have maximum kinetic energy, but near the oscillation nodes the kinetic energy is minimum. The distance between the ranges with maximum particle density (nodes) is the half of the oscillation mode wave length. As consequence, the distribution of the medium particle density will be fractal and exactly the same (isomorphism) as the distribution of the spectral density.

In the phases of spectral compression, where the spectral density increases, in the case of approach to any node arises a particle fusion trend, but in the phases of spectral decompression, where the spectral density decreases, in the case of distance from any node arises a particle dispersion trend. Logarithmic fractal change of spectral compression and decompression generates a logarithmic fractal change of high and low density structure areas inside the medium.

Resonant oscillations can be understood as the most probable forming-mechanism of fractal structures in nature, because the energy efficiency of resonant oscillations is very high.

In the works “About continued fractions” (1737) and “About oscillations of a string” (1748) Leonhard Euler [10] formulated tasks, the solution employed several generations of mathematicians the following 200 years. Euler investigated natural oscillations, based on a model of a massless flexible string with a finite or infinite set of similar pearls. Based on this task d’Alembert developed an intergration method of linear differencial equation systems. Daniel Bernoulli formulated the theorem, that the solution of the problem of the natural oscillations of a string can be represented as trigonometric series, what starts a discussion between Euler, d’Alembert and Bernoulli, and continued several decades.

Later Lagrange showed how can be realised the transition from the solution of the problem of the set with pearls string oscillations to the solution of the oscillations of a homogeneous string. In 1822 Fourier solved this task completely.

Though, big problems arisen with oscillations of strings with a finite set of different pearls. This task leads to functions with gaps. After 1893 Stieltjes [11] investigated such functions and found an integration method, what leads to continued fractions. But only in 1950 Gantmacher and Krein found the general solution of Euler’s task about natural oscillations of a set with pearls string. Gantmacher and Krein interpreted the stretched string between the pearls as a broken line, what opened them a fractal vision of the problem. In the work „Oscillation matrixes, oscillation cores and low oscillations of mechanical systems” Gantmacher and Krein [12] showed that Stieltjes continued fractions are solutions of the Euler-Lagrange equation for low amplitude oscillations of chain systems. These continued fractions generate fractal spectra. Within the fifties and sixties the development of continued fraction analysis methods of oscillation processes in chain systems reaches a highlight. In 1950 Oskar Perron [13] publicated the book “The continued fraction theory”. Achieser [14] investigated continued fractions in the work “The classic problem of moments and some questions of analysis” (1961). In the book “The continued fraction method” (1955) Terskich [15] generalized this method for analysis of oscillations of branched chain systems. In 1964 Khinchine [16] explained the importance of continued fractions in arithmetics and algebra. The works of Khintchine, Markov, Skorobogatko [17] and other mathematicians allowed the development of efficient addition and multiplication methods for continued fractions.

Based on the continued fraction method, in the following we will show, how one can generate scaling spectral models of natural oscillation processes which are not based on any statements about the nature of the link or interaction between the elements of the oscillating system.

3 Fractal scaling spectral models

Based on the continued fraction method we search the natural oscillation frequencies of a chain system of many similar harmonic oscillators in this form:

$$f = f_0 \exp(S)\,, \tag{2}$$

where f is a natural frequency of a chain system of similar harmonic oscillators, f_0 is the natural frequency of one isolated harmonic oscillator, S is a continued fraction with integer elements:

$$S = \frac{n_0}{z} + \cfrac{z}{n_1 + \cfrac{z}{n_2 + \ddots + \frac{z}{n_i}}}\,. \tag{3}$$

The partial numerator z, the free link n_0 and all partial denominators $n_1, n_2, \ldots, n_i$ are integer numbers: $z, n_0, n_i \in Z, i = \overline{1, \infty}$. The present paper follows the Terskich definition of a chain system (Terskich, p. 8) where the interaction between the elements proceeds only in their movement direction. In this connection we understand the concept “spectrum” as a discrete distribution or set of natural oscillation frequencies.

Spectra (2) are not only logarithmic-invariant, but also fractal, because the discrete hyperbolic distribution of natural frequencies repeats itself on each spectral level $i = 1, 2, \ldots$

Every continued fraction (3) with a partial numerator $z \neq 1$ can be changed into a continued fraction with $z = 1$.

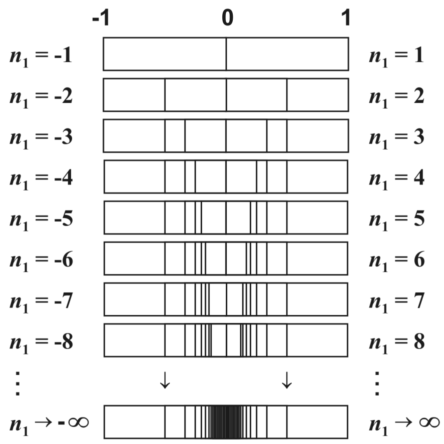

Fig. 2:

For this one can use the Euler equivalent transformation (Skorobogatko, p. 12) and present continued fractions (3) in the canonical form. With the help of the Lagrange [18] transformation (Perron, §40) every continued fraction with integer partial denominators can be represented as a continued fraction with natural partial denominators, what is allways convergent (Khintchine, §4). In this paper we will investigate spectra (2) which are generated by convergent continued fractions (3).

Every infinite continued fraction is irrational, and every irrational number can be represented in precisely one way as an infinite continued fraction (Khintchine, §5). An infinite continued fraction representation for an irrational number is useful because its initial segments provide the best possible rational approximations to the number (Khintchine, §6). These rational numbers are called the convergents of the continued fraction. This last property is quite important, and is not true of the decimal representation. The convergents are rational and therefore they generate a discrete spectrum. Furthermore we investigate continued fractions (3) with a finite quantity of layers $i = \overline{1,k}$ which generate discrete spectra. In the logarithmic representation each natural oscillation frequency can be written down as a finite set of integer elements of the continued fraction (3):

$$\ln(f/f_0) = \frac{n_0}{z} + \cfrac{z}{n_1 + \cfrac{z}{n_2 + \ddots + \frac{z}{n_k}}} = \left[z, n_0, n_1, n_2, \ldots, n_k\right]. \tag{4}$$

Figure 2 shows the generation process of such fractal spectrum for $z = 1$ on the first layer $i = k = 1$ for $|n_1| = 1, 2, 3, \ldots$ and $n_0 = 0$ (logarithmic representation).

The partial denominators n_1 run through positive and negative integer values. Maximum spectral density ranges automatically arise on the distance of 1 logarithmic units, where $n_0 = 0, 1, 2, \ldots$ and $|n_1| \to \infty$. Figure 3 shows the spectrum on the first layer $i = k = 1$ for $|n_1| = 1, 2, 3, \ldots$ and $|n_0| = 0, 1, 2, \ldots$ (logarithmic representation):

Fig. 3:

The more layers $i = 1, 2, 3, \ldots$ are calculated, the more spectral details will be visible. In addition to the first spectral layer, Figure 4 shows the second layer $i = k = 2$ for $|n_2| = 1, 2, 3, \ldots$ and $|n_1| = 2$ (logarithmic representation):

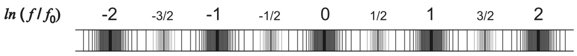

Fig. 4:

On each spectral layer i one can select ranges of relative low spectral density (spectral gaps) and ranges of relative high spectral density (spectral nodes). The highest spectral density corresponds to the nodes on the layer $i = 0$, where $|n_1| \to \infty$. The next (lower) spectral density level corresponds to the nodes on the layer $i = 1$, where $|n_2| \to \infty$, and so on. The largest spectral gaps are between the spectral node ranges on the layer n_0. On the spectral layers $i = 1, 2, 3, \ldots$ the gaps are corresponding smaller.

In 1795 Karl Friedrich Gauss discovered logarithmic scaling invariance of the distribution of prime numbers. Gauss proved, that the quantity of prime numbers $p(n)$ until the natural number n follows the law $p(n) \cong n/\ln(n)$. The equality symbol is correct for the limit $n \to \infty$. The logarithmic scaling distribution is the one and only nontrivial property of all prime numbers.

The free link n_0 and all partial denominators n_1, n_2, n_3, …, n_k are integer numbers and therefore they can be represented as unique products of prime factors. On this base we distinguish spectral classes in dependence on the divisibility of the partial denominators by prime numbers. In addition, we will investigate continued fractions which correspond to the Markov [19] convergence requirement (Skorobogatko, p. 15):

$$|n_i| \geqslant |z_i| + 1. \tag{5}$$

Continued fractions (3) with $z = 1$ and partial denominators divisible by 2 don't generate empty spectral gaps, because the alternating continued fraction [1, 0; +2, −2, +2, −2, …] approximates the number 1 and [1, 0; −2, +2, −2, +2, …] approximates the integer number −1.

Divisible by 3 partial denominators with $z = 2$ build the class of continued fractions (3) what generates the spectrum (4) with the smallest empty spectral gaps. Figure 5 shows fragments of spectra, which were generated by continued fractions (3) with divisible by 2, 3, 4, … partial denominators and corresponding partial numerators $z = 1, 2, 3, \ldots$ on the first layer $i = 1$ for $n_0 = 0$ (logarithmic representation):

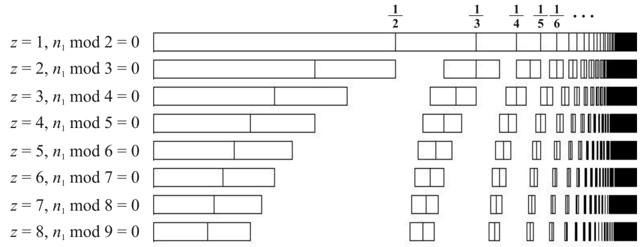

Fig. 5:

Figure 5 shows the spectral nodes on the first layer $i = 1$ and also the borders of the spectral node ranges, so the spectral gaps are visible clearly. The borders of the spectral empty gaps are determined by the following alternating continued fractions ($z \geqslant 1$):

$$\left.\begin{aligned} -1 &= \cfrac{z}{-z-1+\cfrac{z}{z+1+\cfrac{z}{-z-1+\ddots}}} \\ 1 &= \cfrac{z}{z+1+\cfrac{z}{-z-1+\cfrac{z}{z+1+\ddots}}} \end{aligned}\right\} . \qquad (6)$$

More detailled we will investigate the second spectrum of the figure 5, what was generated by the continued fraction (3) with divisible by 3 partial denominators and the corresponding partial numerator $z = 2$. This spectrum is the most interesting one, because with $z = 2$ and $n_i \mod 3 = 0$ starts the generation process of empty gaps. Possibly, that the spectral ranges of these gaps are connected to fundamental properties of oscillation processes.

The partial denominators n_1 run through positive and negative integer values. The maximum spectral density areas arise automatically on the distance of 3/2 logarithmic units, where $n_0 = 3j$, $(j = 0, 1, 2, \dots)$ and $|n_1| \to \infty$. Figure 6 shows the spectrum on the first layer $i = k = 1$ for $|n_1| = 3, 6, 9, \dots$ and $|n_0| = 0, 3, 6, \dots$ (logarithmic representation):

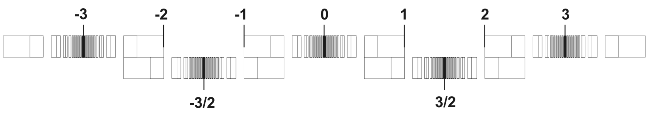

Fig. 6:

The alternating continued fraction [2, 0; +3, −3, +3, −3, . . .] approximates the number 1, but the alternating continued fraction [2, 0; −3, +3, −3, +3, . . .] approximates the number −1. In the consequence the spectral ranges between $|n_1| = 3 - 1$ and $|n_1| = 3 + 1$ are double occupied. The more layers $i = 1, 2, 3, \dots$ are calculated, the more spectral details are visible (see Figure 7).

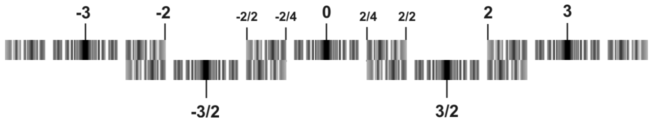

Fig. 7:

Divisible by three free links $|n_0| = 3j$, $(j = 0, 1, 2, \dots)$ of the continued fraction (3) mark the main spectral nodes, partial denominators divisible by three $|n_{i>0}| = 3j$, $(j = 1, 2, \dots)$ mark spectral subnodes. All the other partial denominators $|n_i| \neq 3j$ mark borders of spectral gaps (see Figure 8):

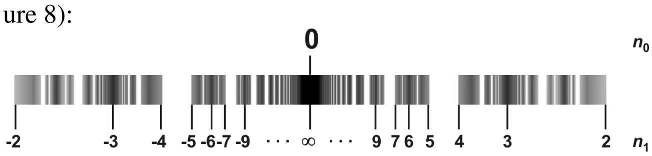

Fig. 8:

4 Local features of fractal scaling spectra and corresponding properties of oscillation processes

In the spectral node ranges, where the spectral density reachs local maximum, the resonance frequencies are distributed maximum densely, so that near a spectral node almost each frequency is a resonance frequency. The energy efficiency of resonant oscillations is very high. Therefore, if a frequency of an oscillation process is located near a node of the fractal spectrum (4), the process energy efficiency (degree of effectiveness) should be relative high. The highest process energy efficiency corresponds to the nodes on the layer $i = 0$. Near the spectral nodes on the layers $i = 1, 2, \dots$ the process energy efficiency should be corresponding lower. On the other hand, if a frequency of an oscillation process is located in a gap of the fractal spectrum (4), the process energy efficiency should be relative low. In the centre of a spectral node the spectral compression changes to spectral decompression (or reversed). Therefore the probability of the process trend change increases near a spectral node.

5 Fractal scaling spectral analysis

Based on the fractal scaling model (2) of resonant oscillations of chain systems one can execute fractal scaling spectral analyses of composite oscillation processes, if the connected oscillators are quite similar.

Corresponding to the logarithmic representation (5) the fractal scaling spectral analysis consists of the following steps:

1. Divide the lowest measured frequency f_{min} and the highest measured frequency f_{max} of an oscillating chain system by the resonance frequency f_0 of one isolated element of the chain system and calculate the natural logarithms $X_{min} = \ln(f_{min}/f_0)$ and $X_{max} = \ln(f_{max}/f_0)$;

2. Use the Euclid's algorithm to find the free links n_0 and partial denominators n_1, n_2, ... of the corresponding to X_{min} and X_{min} continued fractions and determine the location of X_{min} and X_{min} in the spectrum (5);
3. Determine the highest/lowest spectral density ranges of the spectrum (5) between X_{min} and X_{min} which correspond to important properties of the composite oscillation processes;
4. Use the formula (4) to calculate the corresponding frequency ranges.

The fractal scaling spectral analysis is able to define following properties of of composite oscillation processes: turbulence probability, fluctuation probability, resonance probability, stability and sensibility.

6 Resume

The presented model is not based on any statements about the nature of the link or interaction between the elements of the oscillating chain system. Therefore the model statements are quite generally, what opens a wide field of possible applications. Based on the presented model one can use scaling spectral analyses of composite oscillation processes to find out spectral ranges where the process energy efficiency is relative high or low. Possibly, the scaling spectral analysis could be usefull not only in mechanical engineering, but also in nuclear physics and astrophysics.

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