The Physics of Transcendental Numbers

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The difference between rational, irrational algebraic and transcendental numbers is not only a mathematical task, but appears to be a stability criterion in complex dynamic systems. This paper introduces an approach to study the physical consequences of arithmetic properties of real numbers being ratios of measured quantities. This approach allows reformulating and resolving some unsolved tasks in particle physics, astrophysics and cosmology.

Introduction

Natural systems are highly complex and at the same time they impress us with their lasting stability. For instance, the solar system hosts at least 800 thousand orbiting each other bodies. If numerous bodies are gravitationally bound to one another, classic models predict long-term highly unstable states [1, 2] that contradict the physical reality in the solar system. In the last century, advanced models [3–7] were developed, which explain basic features of the solar system formation. However, many metric characteristics of the solar system they do not predict. The problem is that Kepler's laws, the Newton law of gravitation and the Einstein field equations allow for an infinite diversity of orbits.

In reality, however, planets in the extrasolar systems Trappist 1, Kepler 20 and many others have nearly the same orbits as some moons of Jupiter, Saturn, Uranus and Neptune [8]. Why they prefer similar orbits if there are infinite possibilities? Up to now, there have not been sufficiently convincing explanations why the solar system has installed the orbital periods 87.97 days (Mercury), 224,70 days (Venus), 365.25 days (Earth), 686.97 days (Mars), 4.60 years (Ceres), 11.87 years (Jupiter), 29.46 years (Saturn), 84.02 years (Uranus), 164.80 years (Neptune) and 248.00 years (Pluto). In conventional models, they appear as to be accidental.

Furthermore, celestial mechanics does not know any law concerning the periods of planetary rotation. Though, if the periods of rotation are accidental, why then have the Moon and the Sun similar periods of rotation? Why have the Earth, Mars and the planetoid Eris similar periods of rotation? Why have Jupiter, Saturn and the planetoid Ceres similar periods of rotation?

Not only orbital and rotational periods, but also the Earth axial precession cycle (25,770 years), the obliquity variation cycle (41,000 years) as well as the apsidal precession cycle and the orbital eccentricity cycle (both 112,000 years) appear as to be accidental. And this isn't just a shortcoming of astrophysics only.

In particle physics, bosons are considered to have no rest mass, and there are no convincing explanations why the W/Z-bosons must be 90 times as massive as the proton. A rough shortcoming of the Higgs-mechanism of particle mass generation is that the origin of the Higgs-mass itself is not elaborated and this leads to a vicious circle.

Furthermore, there is no convincing explanation why the proton-to-electron mass ratio must be close to 1836 and why these fermions are stable.

Of course, in the standard model, the electron is stable because it is the least massive particle with non-zero electric charge. Its decay would violate charge conservation. Actually, this answer only readdresses the question: What causes then the stability of the elementary electric charge? In the same model, the proton is stable, because it is the lightest baryon and the baryon number is conserved. However, also this answer only readdresses the question: Why then is the proton the lightest baryon? To answer this question, the standard model introduces quarks which violate the conservation of the integer elementary electric charge.

Measurements of the cosmic microwave background radiation (CMBR) are critical to cosmology, since any proposed model of the universe must explain it. However, in Big Bang cosmology, its current average temperature of 2.725 K appears to be accidental, because CMBR is interpreted as a remnant from an early stage of the observable universe when stars and planets didn't exist yet, and the universe was denser and much hotter.

This paper introduces an approach that considers arithmetic properties of the measured ratios of physical quantities. This approach allows not only answering our questions above, but also reformulating and resolving some unsolved tasks in paticle physics, astrophysics and cosmology.

Methods

Measurement is the source of data that allow us developing and proofing theoretical models of the reality. The result of a measurement is the ratio of two physical quantities where one of them is the reference quantity called unit of measurement. In general, this ratio is a real value that can approximate a rational, irrational algebraic or transcendental number.

In [9] we have shown that the difference between rational, irrational algebraic and transcendental numbers is not only a mathematical task, but it is also an essential aspect of stability in complex systems. For instance, integer and rational frequency ratios provide resonance interaction that can destabilize a system.

With reference to the solar system and its stability, we may therefore expect that the ratio of any two orbital periods should be not rational. However, it is not so simple to clarify the type of number a measured ratio approximates. In general, there is no possibility to know it for sure. For example, how can we find out if the Venus-to-Earth orbital period ratio approximates a rational, irrational algebraic or transcendental number?

From the first impression, the obtained value 0.615 seems to be a rational number, but higher resolution data [10] deliver more digits, for example 0.615198 years = 224.701 days = 224 days, 16 hours and 49 minutes. Indeed, also this value is an average. In reality, the sidereal orbital period of Venus is not constant, but varies between 224.695 days = 0.61518 years and 224.709 days = 0.61522 years. According to classic models, that is due to perturbations from other planets, mainly Jupiter and Earth. As well, the orbital period of the Earth is not constant, but shows cyclic variations in the duration up to 7 minutes [11]. However, several authors [12, 13] have suggested that the Venus-to-Earth orbital period ratio coincides with 8/13 approximating the golden section $\phi = (\sqrt{5}-1)/2 = 0.618\ldots$ that is an irrational algebraic number.

It is remarkable that approximation interconnects all types of real numbers – rational, irrational algebraic and transcendental. In 1950, the mathematician Khinchin [14] made an important discovery: He could demonstrate that continued fractions deliver biunique (one-to-one) representations of all real numbers, rational and irrational. Whereas infinite continued fractions represent irrational numbers, finite continued fractions represent always rational numbers. In this way, any irrational number can be approximated by finite continued fractions, which are the convergents and deliver always the nearest and quickest rational approximation.

It is notable that the nearest rational approximation of an irrational number by a finite continued fraction is not a task of computation, but only an act of termination of the fractal recursion. For example, the golden number $\phi = (\sqrt{5}+1)/2$ has a biunique representation as simple continued fraction:

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}.$$

To save space, in the following we use square brackets to write down continued fractions, for example the golden number $\phi = [1; 1, 1, \ldots]$. So long as the sequence of denominators is considered as infinite, this continued fraction represents the irrational number ϕ. If we truncate the continued fraction, the sequence of denominators will be finite and we get a convergent that is always the nearest rational approximation of the irrational number ϕ.

Let's see how it works. Increasing always the length of the continued fraction, we obtain a sequence of rational approximations of ϕ, from the worst to always better and better ones (see Table 1).

Figure 1 demonstrates the process of step by step approximation. As we can see, the rational approximations oscillate around the eigenvalue ϕ of the continued fraction that is shown as dotted line. With every step the approximation comes closer and closer to ϕ, never reaching it and describing a damped asymptotic oscillation around ϕ.

In 1950 Gantmacher and Krein [15] have demonstrated that continued fractions are solutions of the Euler-Lagrange equation for low amplitude harmonic oscillations in simple chain systems. Terskich [16] generalized this method for the analysis of oscillations in branched chain systems. The continued fraction method can also be extended to the analysis of chain systems of harmonic quantum oscillators [17].

The rational approximations of the golden number ϕ are always ratios of neighboring Fibonacci numbers – the elements of the recursive sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... where the sum of two neighbors always yields the following number [18].

As we can see, only the 10$^{\text{th}}$ approximation gives the cor-

Table 1: Approximations of the irrational number ϕ.

[1] = 1
[1; 1] = 2
[1; 1, 1] = 3/2 = 1.5
[1; 1, 1, 1] = 5/3 = $1.\overline{66}$
[1; 1, 1, 1, 1] = 8/5 = 1.6
[1; 1, 1, 1, 1, 1] = 13/8 = 1.625
[1; 1, 1, 1, 1, 1, 1] = 21/13 = $1.\overline{615384}$
[1; 1, 1, 1, 1, 1, 1, 1] = 34/21 = $1.\overline{619047}$
[1; 1, 1, 1, 1, 1, 1, 1, 1] = 55/34 = $1.6\overline{1764705882352941}$
[1; 1, 1, 1, 1, 1, 1, 1, 1, 1] = 89/55 = $1.6\overline{18}$

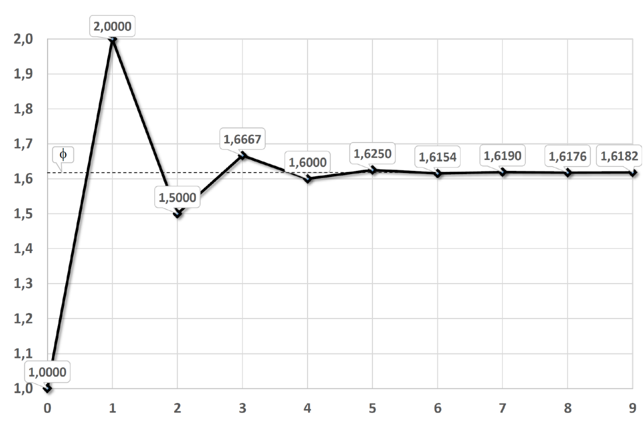

Fig. 1: The approximation steps 0 – 9 of the golden number $\phi = 1.618\ldots$ (dotted line) by continued fraction [1; 1, 1, ...].

rect third decimal of ϕ. The approximation process is very slow because of the small denominators. In fact, the denominators in the continued fraction of ϕ are the smallest possible and consequently, the approximation speed is the lowest possible. The golden number ϕ is therefore treated as the "most irrational" number in the sense that a good approximation of ϕ by rational numbers cannot be given with small quotients.

On the contrary, transcendental numbers can be approximated exceptionally well by rational numbers, because their continued fractions contain large denominators and can be truncated with minimum loss of precision. For instance, the simple continued fraction of the number $\pi = 3.1415927\ldots = [3;\, 7,\, 15,\, 1,\, 292,\, \ldots]$ delivers the following sequence of rational approximations:

$[3] = 3$
$[3;\, 7] = 3.\overline{142857}$
$[3;\, 7,\, 15] = 3.\overline{14150943396226}$
$[3;\, 7,\, 15,\, 1] = 3.1415929\ldots$

We can see that the 2$^{\text{nd}}$ approximation delivers the first 2 decimals correctly, and the 4$^{\text{th}}$ approximation shows already 6 correct decimals.

Much like the continued fraction of the golden number ϕ contains only the number 1, a prominent continued fraction [19] of Euler's number contains all natural numbers as denominators and numerators, forming an infinite fractal sequence of harmonic intervals:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cdots}}}}.$$

As Euler's number is transcendental, it can also be represented as a continued fraction with quickly increasing denominators:

$$e = 1 + \cfrac{2}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cdots}}}}.$$

In this way, already the 4$^{\text{th}}$ approximation delivers the first 3 decimals correctly and returns in fact the rounded Euler's number $e = 2.71828\ldots$ of 5 decimals' resolution:

1
3
$2.\overline{714285}$
$2.7183\ldots$

This special arithmetic property of continued fractions [20] of transcendental numbers has the consequence that transcendental numbers are distributed near by rational numbers of small quotients or close to integers, like $e^3 = 20.08\ldots$ or $e^{4.5} = 90.01\ldots$. This can create the impression that complex systems like the solar system provide ratios of physical quantities that approximate rational numbers. More likely, they approximate transcendental numbers, which are located close to rational numbers.

Namely, transcendental numbers define the preferred ratios of quantities which avoid destabilizing resonance interaction [9]. In this way, they sustain the lasting stability of periodic processes in complex dynamic systems. At the same time, a good rational approximation can be induced quickly, if the system temporarily requires local resonance interaction. Though, algebraic irrational numbers like $\sqrt{2}$ or the golden number ϕ do not compellingly prevent resonance, because they can be transformed into integer or rational numbers by multiplication.

Among all transcendental numbers, Euler's number $e = 2.71828\ldots$ is unique, because its real power function e^x coincides with its own derivatives. In the consequence, Euler's number allows inhibiting resonance interaction regarding any interacting periodic processes and their derivatives. Because of this unique property of Euler's number, complex dynamic systems tend to establish relations of quantities that coincide with values of the natural exponential function e^x for integer and rational exponents x.

Therefore, we expect that periodic processes in real systems prefer frequency ratios close to Euler's number and its rational powers. Consequently, the logarithms of the frequency ratios should be close to integer 1, 2, 3, 4, ... or rational values $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ In [21] we exemplified our hypothesis in particle physics, astrophysics, cosmology, geophysics, biophysics and engineering.

Thanks to Khinchin's [14] discovery, any real number has a biunique representation as a continued fraction. Now let's apply this to the real argument x of the natural exponential function e^x itself:

$$x = [n_0; n_1, n_2, \ldots, n_k]. \tag{1}$$

All denominators $n_1, n_2, \ldots, n_k$ of the continued fraction including the free link n_0 are integer numbers. All numerators equal 1. The length of the continued fraction is given by the number k of layers.

The canonical form (all numerators equal 1) does not limit our conclusions, because every continued fraction with partial numerators different from 1 can be transformed into a canonical continued fraction using the Euler equivalent transformation [22]. With the help of the Lagrange [23] transformation, every continued fraction with integer denominators can be represented as a continued fraction with natural denominators that is always convergent [24].

Now we are going to study the fractal distribution of the rational eigenvalues of the finite continued fractions (1). The first layer is given by the truncated after n_1 continued fraction:

$$x = [n_0; n_1] = n_0 + \frac{1}{n_1}.$$

For the beginning we take $n_0 = 0$. The denominators n_1 follow the sequence of integer numbers ±1, ±2, ±3 etc. The second layer is given by the truncated after n_2 continued fraction:

$$x = [n_0; n_1, n_2] = n_0 + \cfrac{1}{n_1 + \cfrac{1}{n_2}}.$$

Figure 2 shows the first and the second layer in comparison. As we can see, reciprocal integers $\pm 1/2, \pm 1/3, \pm 1/4, \ldots$ are the attractor points of the distribution. In these points, the distribution density always reaches a local maximum. Whole numbers $0, \pm 1, \ldots$ are the main attractors of the distribution.

Now let's remember that we are observing the fractal distribution of rational values $x = [n_0; n_1, n_2, \ldots, n_k]$ of the real argument x of the natural exponential function e^x. What we see is the fractal distribution of transcendental numbers of the type $\exp([n_0; n_1, n_2, \ldots, n_k])$ on the natural logarithmic scale. Near integer exponents the distribution density of these transcendental numbers is maximum.

Consequently, for integer exponents x, the natural exponential function e^x defines attractor points of transcendental numbers and create islands of stability.

Figure 2 shows that these islands are not points, but ranges of stability. Integer exponents $0, \pm 1, \pm 2, \pm 3, \ldots$ are attractors which form the widest ranges of stability. Half exponents $\pm 1/2$ form smaller islands, one third exponents $\pm 1/3$ form the next smaller islands and one fourth exponents $\pm 1/4$ form even smaller islands of stability etc.
For rational exponents, the natural exponential function is always transcendental [25]. Increasing the length of the continued fraction (1), the density of the distribution of transcendental numbers of the type $\exp([n_0; n_1, n_2, \ldots, n_k])$ is increasing as well. Nevertheless, their distribution is not homogeneous, but fractal. Applying continued fractions and truncating them, we can represent the real exponents x of the natural exponential function e^x as rational numbers and make visible their fractal distribution.

Here I would like to underline that the application of continued fractions doesn't limit the universality of our conclusions, because continued fractions deliver biunique representations of all real numbers including transcendental. Therefore, the fractal distribution of transcendental eigenvalues of the natural exponential function e^x of the real argument x, represented as continued fraction, is an inherent characteristic of the number continuum. This characteristic we call the Fundamental Fractal [26].

Fig. 2: The Fundamental Fractal – the fractal distribution of transcendental numbers of the type e^x with $x = [n_0; n_1, n_2, \ldots, n_k]$ on the natural logarithmic scale for $k = 1$ (first layer above) and for $k = 2$ (second layer below) in the range $-3/2 \leqslant x \leqslant 3/2$.

In physical applications, the natural exponential function e^x of the real argument x is the ratio of two physical quantities where one of them is the reference quantity called unit of measurement. Therefore, we can rewrite the equation (1):

$$\ln(X/Y) = [n_0; n_1, n_2, \ldots, n_k], \tag{2}$$

where X is the measured physical quantity and Y the unit of measurement.

In this way, the natural exponential function e^x of the rational argument $x = [n_0; n_1, n_2, \ldots, n_k]$ generates the set of preferred ratios X/Y of quantities which avoid destabilizing resonance and in this way, provide the lasting stability of real systems regardless of their complexity. This is a very powerful conclusion, as we will see in the following.

Results

Now let's apply this result to our first example of the Venus-to-Earth orbital period ratio. In this case, X = 224.701 days and Y = 365.256363 days. Following (2) we calculate the natural logarithm ln (X/Y):

$$\ln\left(\frac{Venus\ orbital\ period}{Earth\ orbital\ period}\right) = \ln\left(\frac{224.701}{365.256363}\right) = -0.49.$$

We can see that this logarithm is close to $-1/2$. The deviation is only 0.01. In accordance with (2), $n_0 = 0$ and $n_1 = 2$. Consequently, the Venus-to-Earth orbital period ratio is close to an attractor point of the Fundamental Fractal, the center of a local island of stability.

In fact, the ratios of the orbital periods in the solar system approximate Euler's number and its rational powers [9]. Obviously, in this way, the solar system can ovoid destabilizing resonance of the orbital motions and reach lasting stability. For instance, Saturn's sidereal orbital period [27] equals 10759.22 days, that of Uranus is 30688.5 days. The natural logarithm of the ratio of their orbital periods is close to 1:

$$\ln\left(\frac{Uranus\ orbital\ period}{Saturn\ orbital\ period}\right) = \ln\left(\frac{30688.5}{10759.22}\right) = 1.05.$$

Jupiter's sidereal orbital period equals 4332.59 days, that of the planetoid Ceres is 1681.63 days. The natural logarithm of the ratio of their orbital periods is also close to 1:

$$\ln\left(\frac{Jupiter\ orbital\ period}{Ceres\ orbital\ period}\right) = \ln\left(\frac{4332.59}{1681.63}\right) = 0.95.$$

Not only neighboring orbits show Euler ratios, but far apart from each other orbits do this as well. Pluto's sidereal orbital period is 90560 days, that of Venus is 224.701 days. The natural logarithm of the ratio of their orbital periods equals 6:

$$\ln\left(\frac{Pluto\ orbital\ period}{Venus\ orbital\ period}\right) = \ln\left(\frac{90560}{224.701}\right) = 6.00.$$

In [8] we have analyzed the orbital periods of the largest bodies in the solar system including the moon systems of Jupiter, Saturn, Uranus and Neptune, as well as the exoplanetary systems Trappist 1 and Kepler 20. In the result we can assume that the stability of all these orbital systems is given by the transcendence of Euler's number and its rational powers.

The most stable systems we know are of atomic scale. Because of their exceptional stability, proton and electron form stable atoms, the structural elements of matter. The lifespans of the proton and electron surpass everything that is measurable, exceeding 10^{30} years. The proton-to-electron ratio 1836.152674 is considered as fundamental physical constant [28] and it has the same value for their rest energies and rest masses, frequencies and wavelengths. The natural logarithm is close to seven and a half:

$$\ln(1836.152674) = 7.515427\ldots \simeq 6 + \frac{3}{2}.$$

This result suggests the assumption that the stability of the proton and electron comes from the number continuum, more specifically, from the transcendence of Euler's number and its rational powers. Already in the eighties the scaling exponent 3/2 was found in the distribution of particle masses by Valery Kolombet [29]. Applying hyperscaling [26] by Euler's number (tetration), we get the next approximation of the logarithm of the proton-to-electron ratio:

$$6 + \frac{e^e}{10} = 7.515426\ldots$$

We suppose that hyperscaling by Euler's number causes the exceptional stability of proton and electron.

In [17] we have analyzed the mass distribution of hadrons, mesons, leptons, the W/Z and Higgs bosons and proposed scaling by Euler's number and its roots as model of particle mass generation [30]. In this model, the $W^{\pm}$-boson mass 80385 MeV/c^2 and the Z^0-boson mass 91188 MeV/c^2 appear as the 12 times scaled up electron rest mass 0.511 MeV/c^2:

$$\ln\left(\frac{W^{\pm}}{electron}\right) = \ln\left(\frac{80385}{0.511}\right) = 11.97.$$

$$\ln\left(\frac{Z^0}{electron}\right) = \ln\left(\frac{91188}{0.511}\right) = 12.09.$$

Expected, the square root of Euler's number defines the next island of stability – in fact, the corresponding state of matter was discovered in 2012 and interpreted [31] as Higgs-boson H^0 with the rest mass 125.18 GeV/c^2:

$$\ln\left(\frac{H^0}{electron}\right) = \ln\left(\frac{125180}{0.511}\right) = 12.41.$$

Euler's number and its rational powers are universal scaling factors that inhibit resonance and in this way, stabilize periodic processes bound in a chain system. This approach we call Global Scaling [21]. The rest energy of the proton can be seen as the $6+\frac{3}{2}$ times scaled up rest energy of the electron. In the same way, Pluto's orbital period can be seen as the 6 times scaled up by Euler's number orbital period of Venus or as the 3 times scaled up by Euler's number orbital period of Jupiter. Here it is important to understand that only scaling by Euler's number and its rational powers inhibits resonance interaction and provides lasting stability of bound processes and allows for the formation of stable atoms or stable planetary systems, for instance.

Now we could ask the question: Starting with the electron oscillation period, if we continue to scale up always multiplying by Euler's number, will we meet the orbital period, for instance, of Jupiter?

Actually, it is so. If we multiply the electron oscillation period 66 times by Euler's number, we meet exactly the orbital period of Jupiter:

$$\ln\left(\frac{T_{Jupiter\ orb}}{\tau_{electron}}\right) = \ln\left(\frac{3.7434 \cdot 10^8\ \text{s}}{8.093 \cdot 10^{-21}\ \text{s}}\right) = 66.00.$$

Jupiter's orbital period $T_{Jupiter\ orb}$ = 4332.59 days = 3.7434×10^8 s. The oscillation period of the electron $\tau_{electron}$ derives from its rest energy $E_{electron}$ = 0.511 MeV:

$$\tau_{electron\ angular} = \hbar / E_{electron} = 1.288 \times 10^{-21}\text{s},$$

$$\tau_{electron} = 2\pi \cdot \tau_{electron\ angular} = 8.093 \times 10^{-21}\text{s}.$$

$\hbar$ is the reduced Planck constant. Data taken from [28]. Similarly, the oscillation period of the proton τ_{proton} derives from its rest energy E_{proton} = 938.272 MeV:

$$\tau_{proton\ angular} = \hbar / E_{proton} = 7.015 \times 10^{-25}\text{s},$$

$$\tau_{proton} = 2\pi \cdot \tau_{proton\ angular} = 4.408 \times 10^{-24}\text{s}.$$

Within our approach, electron and proton define two complementary classes of stability in the sense of the avoidance of destabilizing resonance. Here and in the following, we use the letter E for electron stability and the letter P for proton stability. In accordance with (2), we use rectangle brackets for continued fractions. For example, $E[66]$ means the main attractor 66 of electron stability. In the solar system, this attractor stabilizes the orbital period of Jupiter.

The main attractor $E[63]$ stabilizes the orbital period of Venus. The siderial orbital period of Venus $T_{Venus\ orb}$ equals 224.701 days = 1.9414×10^7 s:

$$\ln\left(\frac{T_{Venus\ orb}}{\tau_{electron}}\right) = \ln\left(\frac{1.9414 \times 10^7\ \text{s}}{8.093 \times 10^{-21}\ \text{s}}\right) = 63.04 = E[63].$$

Not only the orbits of planets and planetoids, but also the orbits of moons are stabilized by the Fundamental Fractal (2).

For example, the main attractor $E[61]$ stabilizes the orbital period $T_{Moon\ orb}$ = 27.321661 days = 2.36059×10^6 s of the Moon:

$$\ln\left(\frac{T_{Moon\ orb}}{\tau_{electron}}\right) = \ln\left(\frac{2.36059 \times 10^6\ \text{s}}{8.093 \times 10^{-21}\ \text{s}}\right) = 60.94 = E[61].$$

The attractor $E[62]$ stabilizes the orbital period of Saturn's moon Iapetus $T_{Iapetus\ orb}$ = 79.3215 days = 6.8534×10^6 s:

$$\ln\left(\frac{T_{Iapetus\ orb}}{\tau_{electron}}\right) = \ln\left(\frac{6.8534 \times 10^6\ \text{s}}{8.093 \times 10^{-21}\ \text{s}}\right) = 62.00 = E[62].$$

As well, it is not surprising that Ceres, the largest body of the main asteroid belt, orbits the Sun close to a main attractor. The orbital period of Ceres $T_{Ceres\ orb}$ equals 1681.63 days = 1.4529×10^8 s:

$$\ln\left(\frac{T_{Ceres\ orb}}{\tau_{electron}}\right) = \ln\left(\frac{1.4529 \times 10^8\ \text{s}}{8.093 \times 10^{-21}\ \text{s}}\right) = 65.05 = E[65].$$

Now let us analyze some rotational periods. Although the rotation of Venus is retrograde, its period $T_{Venus\ rot}$ = 5816.667 hours = 2.094×10^7 s is close to the main attractor $E[65]$:

$$\ln\left(\frac{T_{Venus\ rot}}{\tau_{electron\ angular}}\right) = \ln\left(\frac{2.094 \times 10^7\ \text{s}}{1.288 \times 10^{-21}\ \text{s}}\right) = 64.96 = E[65].$$

As well, the full rotational period of the Sun $T_{Sun\ rot}$ = 34.3 days = 2.9635×10^6 s fits with a main attractor:

$$\ln\left(\frac{T_{Sun\ rot}}{\tau_{electron\ angular}}\right) = \ln\left(\frac{2.9635 \times 10^6\ \text{s}}{1.288 \times 10^{-21}\ \text{s}}\right) = 63.00 = E[63].$$

As we have seen, the main attractor $E[63]$ stabilizes the rotational period of the Sun as well as the orbital period of Venus. From this, directly follows:

$$T_{Venus\ orb} = 2\pi \cdot T_{Sun\ rot}$$

Although π is transcendental, its real power function π^x does not coincide with its own derivatives. Therefore, π cannot inhibit resonance interaction regarding the derivatives of periodic processes, but it does not violate the transcendence [32] of Euler's number. Within our approach, 2π connects stable rotation with stable orbital motion.

In addition, the main attractor $E[65]$ stabilizes the orbital period of Ceres as well as the rotational period of Venus. From this, directly follows:

$$T_{Ceres\ orb} = 2\pi \cdot T_{Venus\ rot}$$

Obviously, preferred rotational periods are not accidental, but follow the Fundamental Fractal (2) and are connected by 2π with stable, avoiding resonance orbital periods.

Within our approach, the approximation level of an attractor of stability indicates evolutionary trends. For example, the orbital period of Venus must still decrease for reaching the center of $E[63]$. On the contrary, the orbital period of the Moon must still increase for reaching the center of $E[61]$. Actually, exactly this is observed [33].

While all the orbital and rotational periods we have analyzed are stabilized by main attractors of electron stability, the rotational period of Mars $T_{Mars\ rot}$ = 24.62278 hours = 88642 s approximates a main attractor of proton stability:

$$\ln\left(\frac{T_{Mars\ rot}}{\tau_{proton\ angular}}\right) = \ln\left(\frac{88642\ \text{s}}{7.015 \times 10^{-25}\ \text{s}}\right) = 67.01 = P[67].$$

The rotational period of the Earth $T_{Earth\ rot}$ = 23.934 hours = 86164 s approximates the same attractor $P[67]$:

$$\ln\left(\frac{T_{Earth\ rot}}{\tau_{proton\ angular}}\right) = \ln\left(\frac{86164\ \text{s}}{7.015 \times 10^{-25}\ \text{s}}\right) = 66.98 = P[67].$$

This means that the main attractor $P[67]$ stabilizes the rotational periods of Mars and Earth. Furthermore, the attractor $P[71]$ stabilizes the orbital period $T_{Earth\ orb}$ = 365.25636 days = 3.1558×10^7 s of the Earth:

$$\ln\left(\frac{T_{Earth\ orb}}{\tau_{proton}}\right) = \ln\left(\frac{3.1558 \times 10^7\ \text{s}}{4.408 \times 10^{-24}\ \text{s}}\right) = 71.05 = P[71].$$

Obviously, the Earth's orbital eccentricity variation cycle $T_{Earth\ orb\ ecc} \approx 112,600$ years = 3.5533×10^{12} s is stabilized by the main attractor $E[77]$:

$$\ln\left(\frac{T_{Earth\ orb\ ecc}}{\tau_{electron\ angular}}\right) = \ln\left(\frac{3.5533 \times 10^{12}\ \text{s}}{1.288 \times 10^{-21}\ \text{s}}\right) = 77.00 = E[77].$$

This attractor stabilizes also the Earth's apsidal precession cycle $\approx 112,000$ years. The Earth's orbital inclination variation cycle $T_{Earth\ orb\ inc} \approx 70,000$ years = $2.209 \cdot 10^{12}$ s is stabilized by the attractor $E[76; 2]$:

$$\ln\left(\frac{T_{Earth\ orb\ inc}}{\tau_{electron\ angular}}\right) = \ln\left(\frac{2.209 \times 10^{12}\ \text{s}}{1.288 \times 10^{-21}\ \text{s}}\right) = 76.51 = E[76; 2].$$

The obliquity variation cycle of the ecliptic $T_{Ecliptic\ obliquity} \approx 41,000$ years = 1.2938×10^{12} s is stabilized by the main attractor $E[76]$:

$$\ln\left(\frac{T_{Ecliptic\ obliquity}}{\tau_{electron\ angular}}\right) = \ln\left(\frac{1.2938 \times 10^{12}\ \text{s}}{1.288 \times 10^{-21}\ \text{s}}\right) = 75.99 = E[76].$$

The Earth's axial precession cycle $T_{Earth\ axial\ prec} \approx 25,770$ years = 8.1328×10^{11} s is stabilized by the attractor $E[75; 2]$:

$$\ln\left(\frac{T_{Earth\ axial\ prec}}{\tau_{electron\ angular}}\right) = \ln\left(\frac{8.1328 \times 10^{11}\ \text{s}}{1.288 \times 10^{-21}\ \text{s}}\right) = 75.52 = E[75; 2].$$

The Earth's axial nutation period $T_{Earth\ axial\ prec}$ = 18.6 years = 5.8696×10^8 s is stabilized by the main attractor $P[74]$:

$$\ln\left(\frac{T_{Earth\ axial\ prec}}{\tau_{proton}}\right) = \ln\left(\frac{5.8696 \times 10^8\ \text{s}}{4.408 \times 10^{-24}\ \text{s}}\right) = 73.97 = P[74].$$

The Chandler wobble of the Earth's axis $T_{Chandler\ wobble} = 433$ days $= 3.741 \times 10^7$ s is stabilized by the main attractor $P[73]$:

$$\ln\left(\frac{T_{Chandler\ wobble}}{\tau_{proton\ angular}}\right) = \ln\left(\frac{3.741 \times 10^7\ \text{s}}{7.015 \times 10^{-25}\ \text{s}}\right) = 73.05 = P[73].$$

As we have seen, within our approach, the current orbital and rotational periods in the solar system do not appear as to be accidental, but correspond with islands of stability defined by Euler's number and its rational powers that allow avoiding destabilizing resonance. This is valid not only for the solar system, but also for exoplanetary systems as we have shown in [8]. Furthermore, our approach explains the durations of the axial precession cycle including the nutation period and the Chandler wobble, the obliquity variation cycle, the orbital inclination variation cycle, the apsidal precession cycle and the orbital eccentricity cycle of the Earth.

In [21] we have shown that the divisibility of their integer logarithms interconnects all the main attractors of electron and proton stability and causes interscalar effects, which stabilize also biophysical periodical processes.

Concluding this overview, I would like to mention that, within our approach, the current average temperature $\text{T}_{\text{CMBR}} = 2.725$ K [34] of the cosmic microwave background radiation (CMBR) does not appear to be accidental. On the contrary, obviously, this process is stable, because its average temperature is close to a main attractor of proton stability:

$$\ln\left(\frac{\text{T}_{\text{CMBR}}}{\text{T}_{\text{proton}}}\right) = \ln\left(\frac{2.725\ \text{K}}{1.0888 \times 10^{13}\ \text{K}}\right) = -29.01 = P[-29].$$

The proton blackbody temperature $\text{T}_{\text{proton}} = E_{proton}/k$ derives from the proton rest energy $E_{proton} = 938.272\, MeV$ and the Boltzmann [28] constant k.

Consequently, the current temperature of the CMBR is not accidental, and it is highly unlikely that this temperature will still decrease.

In [35] we have shown that integer powers of Euler's number define also the ratios of fundamental physical constants. In our approach, this means that the transcendence of Euler's number stabilizes energy-frequency and energy-mass conversions and makes possible the existence of fundamental physical constants. For instance, the 88^{th} power of Euler's number stabilizes the ratio of the speed of light c, the Planck constant $\hbar$, the proton rest mass m_p and the gravitational constant G:

$$\frac{\hbar \cdot c}{G \cdot m_p^2} = e^{88}. \qquad (3)$$

Quantum mechanics only postulates, but does not derive the constancy of the Planck constant as well as general relativity postulates the constancy of the gravitational constant, but does not derive it. Also special relativity postulates, but does not derive the constancy of the speed of light. Up to now, there have not been sufficiently convincing explanations why the speed of light should be constant, why it should have the value 299792458 m/s and why it should be the maximum possible velocity in the universe.

Within our approach, we can derive the speed of light c from other fundamental physical constants stabilized by integer powers of Euler's number. Naturally, the proton is not the only stable particle. The electron is stable as well. Furthermore, the proton-to-electron ratio is stabilized by Euler's number and its rational powers. From this and (3), directly follows that 299792458 m/s is not the maximum speed. Indeed, rational powers of Euler's number define a logarithmically fractal set of stable velocities $c_{n,m}$ which are superluminal for $n > 0$:

$$c_{n,m} = c \cdot e^{n/m}$$

where n, m are integer numbers. In general, the rational exponents are finite continued fractions (1). In [35] we verified the fractal set $c_{n,m}$ of stable subluminal and superluminal velocities on experimental and astrophysical data.

Conclusion

In this paper, we discussed the physical significance of transcendental numbers approximated by ratios of physical quantities. In particular, the transcendence of Euler's number allows avoiding destabilizing resonance interaction in real systems and appears to be a universal criterion of stability.

For instance, Euler's number and its rational powers stabilize the orbital and rotational periods of planets, planetoids and moons in the solar system.

Our approach allows deriving the mass ratios of the fundamental elementary particles electron, proton, $W^{\pm}$, Z^0 and H^0-boson as well as the temperature 2.725 K of the cosmic microwave background from Euler's number and its rational powers. Integer powers of Euler's number stabilize also the ratios of the fundamental physical constants $\hbar$, c, G.

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